3. Circuit implementation of the transfer function

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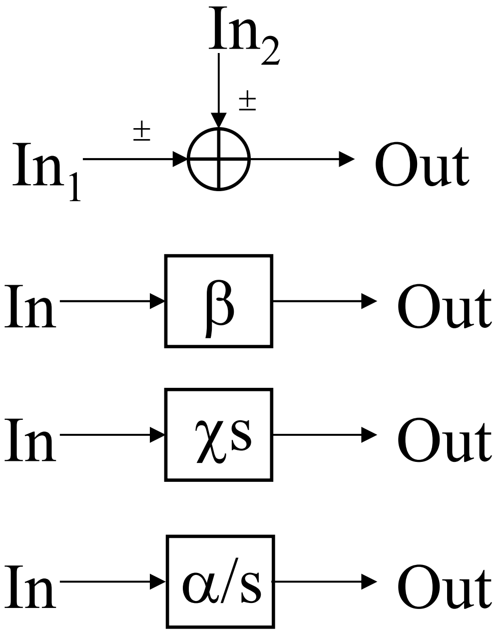
3.1 Block diagram of continuous-time circuit

Linear elements for continuous-time circuit



Adder and subtractor

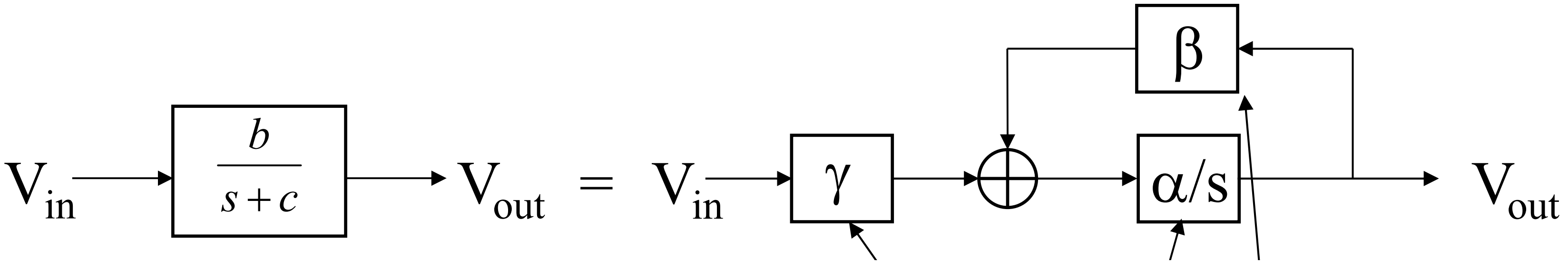
Constant multiplication

Differentiator

Integrator

The parameters β, χ, and α are the gain of an amplifier, a differentiator, and an integrator, respectively.

Block diagram of 1st-order LPF



Transfer function

Block diagram of transfer function

$$\frac{V_{out}}{V_{in}} = \frac{b}{s+c}$$

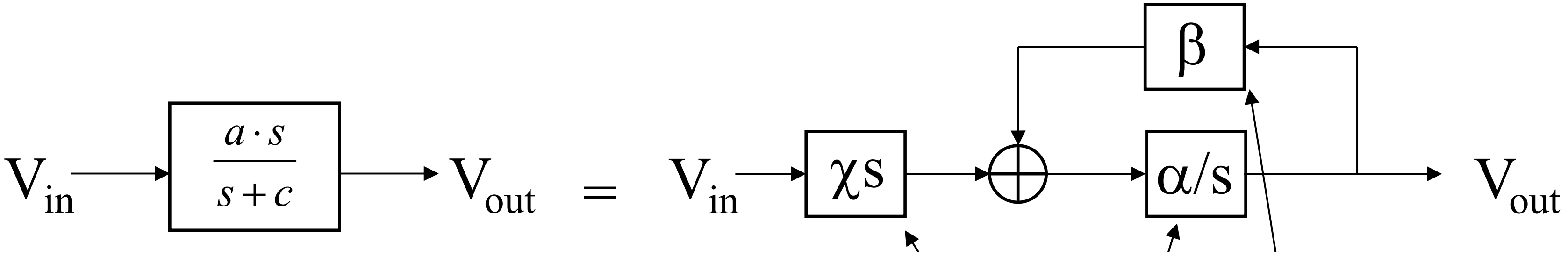
$$(s+c)\cdot V_{out} = b\cdot V_{in}$$

$$V_{out} = \frac{1}{s}(b\cdot V_{in} - c\cdot V_{out}) = \frac{\alpha}{s}(\frac{b}{\alpha}V_{in} - \frac{c}{\alpha}V_{out}) \equiv \frac{\alpha}{s}(\gamma\cdot V_{in} + \beta\cdot V_{out})$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{b}{s+c} = \frac{\alpha\cdot\gamma}{s-\alpha\cdot\beta} \longleftarrow \begin{cases} b = \alpha\cdot\gamma \\ c = -\alpha\cdot\beta \end{cases}$$

NOTE: A feedback of a constant introduces a constant term in a denominator.

Block diagram of 1st-order HPF



Transfer function

Block diagram of transfer function

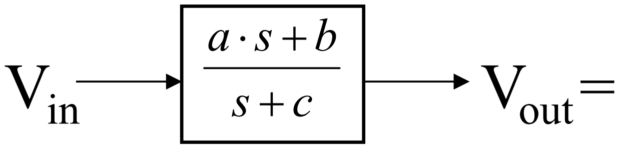
$$\frac{V_{out}}{V_{in}} = \frac{a \cdot s}{s+c}$$

$$(s+c) \cdot V_{out} = a \cdot s \cdot V_{in}$$

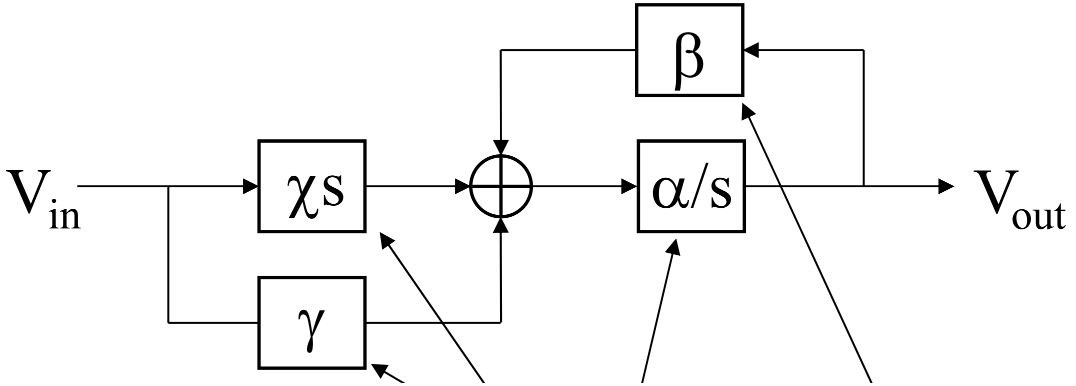
$$V_{out} = \frac{1}{s}(a \cdot s \cdot V_{in} - c \cdot V_{out}) = \frac{\alpha}{s}(\frac{a}{\alpha} s \cdot V_{in} - \frac{c}{\alpha} V_{out}) \equiv \frac{\alpha}{s}(\chi \cdot s \cdot V_{in} + \beta \cdot V_{out})$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{a \cdot s}{s+c} = \frac{\alpha \cdot \chi \cdot s}{s - \alpha \cdot \beta} \longleftarrow \begin{cases} a = \alpha \cdot \chi \\ c = -\alpha \cdot \beta \end{cases}$$

Block diagram of 1st-order transfer function



Transfer function



Block diagram of transfer function

$$\frac{V_{out}}{V_{in}}=\frac{a\cdot s+b}{s+c}$$

$$(s+c)\cdot V_{out}=(a\cdot s+b)\cdot V_{in}$$

$$V_{out}=\frac{1}{s}(a\cdot s\cdot V_{in}+b\cdot V_{in}-c\cdot V_{out})=\frac{\alpha}{s}(\frac{a}{\alpha}s\cdot V_{in}+\frac{b}{\alpha}V_{in}-\frac{c}{\alpha}V_{out})\equiv\frac{\alpha}{s}(\chi\cdot s\cdot V_{in}+\gamma\cdot V_{in}+\beta\cdot V_{out})$$

$$H(s)=\frac{V_{out}}{V_{in}}=\frac{a\cdot s+b}{s+c}=\frac{\alpha\cdot\chi\cdot s+\alpha\cdot\gamma}{s-\alpha\cdot\beta} \longleftarrow \begin{cases} a=\alpha\cdot\chi \\ b=\alpha\cdot\gamma \\ c=-\alpha\cdot\beta \end{cases}$$

Block diagram of 2nd-order LPF 1

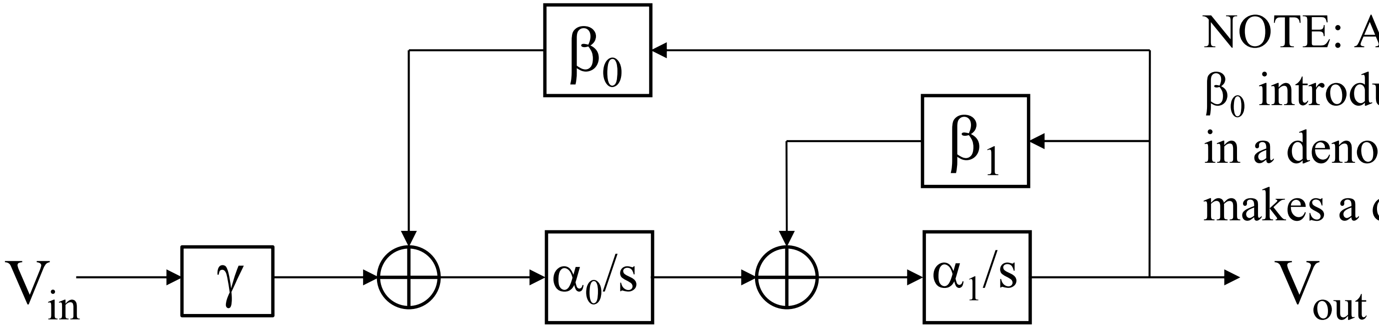
$$\frac{V_{out}}{V_{in}} = \frac{c}{s^2 + d \cdot s + e}$$

$$V_{in} \longrightarrow \boxed{\frac{c}{s^2 + d \cdot s + e}} \longrightarrow V_{out}$$

$$(s^2 + d \cdot s + e) \cdot V_{out} = c \cdot V_{in}$$

$$V_{out} = \frac{1}{s^2}(c \cdot V_{in} - d \cdot s \cdot V_{out} - e \cdot V_{out}) = \frac{1}{s}\{\frac{1}{s}(c \cdot V_{in} - e \cdot V_{out}) - d \cdot V_{out}\}$$

$$= \frac{\alpha_1}{s}\{\frac{\alpha_0}{s}(\frac{c}{\alpha_1 \alpha_0} V_{in} - \frac{e}{\alpha_1 \alpha_0} V_{out}) - \frac{d}{\alpha_1} V_{out}\} \equiv \frac{\alpha_1}{s}\{\frac{\alpha_0}{s}(\gamma \cdot V_{in} + \beta_0 \cdot V_{out}) + \beta_1 \cdot V_{out}\}$$

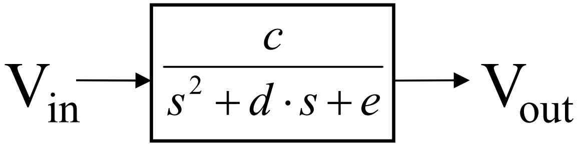


NOTE: A feedback element β_0 introduces a constant term in a denominator, and β_1 makes a design flexibility.

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{c}{s^2 + d \cdot s + e} = \frac{\alpha_0 \cdot \alpha_1 \cdot \gamma}{s^2 - \alpha_1 \cdot \beta_1 \cdot s - \alpha_0 \cdot \alpha_1 \cdot \beta_0} \longleftarrow \begin{cases} c = \alpha_0 \cdot \alpha_1 \cdot \gamma \\ d = -\alpha_1 \cdot \beta_1 \\ e = -\alpha_0 \cdot \alpha_1 \cdot \beta_0 \end{cases}$$

Block diagram of 2nd-order LPF 2

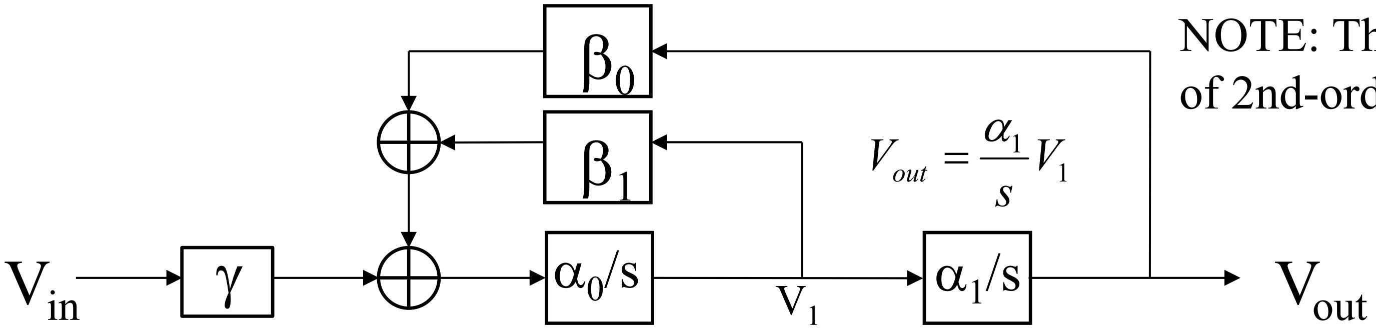
$$\frac{V_{out}}{V_{in}} = \frac{c}{s^2 + d\cdot s + e}$$



$$(s^2 + d\cdot s + e)\cdot V_{out} = c\cdot V_{in}$$

$$V_{out} = \frac{1}{s^2}(c\cdot V_{in} - d\cdot s\cdot V_{out} - e\cdot V_{out}) = \frac{\alpha_1}{s}\frac{\alpha_0}{s}(\frac{c}{\alpha_1\alpha_0}\cdot V_{in} - \frac{d}{\alpha_0}\frac{s}{\alpha_1}V_{out} - \frac{e}{\alpha_1\alpha_0}V_{out})$$

$$= \frac{\alpha_1}{s}\frac{\alpha_0}{s}(\frac{c}{\alpha_1\alpha_0}\cdot V_{in} - \frac{d}{\alpha_0}V_1 - \frac{e}{\alpha_1\alpha_0}V_{out}) \equiv \frac{\alpha_1}{s}\frac{\alpha_0}{s}(\gamma\cdot V_{in} + \beta_1\cdot V_1 + \beta_0\cdot V_{out})$$



$$V_{out} = \frac{\alpha_1}{s}V_1$$

NOTE: This is another solution of 2nd-order LPF.

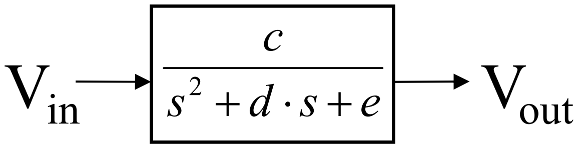
$$H(s) = \frac{V_{out}}{V_{in}} = \frac{c}{s^2 + d\cdot s + e} = \frac{\alpha_0\cdot\alpha_1\cdot\gamma}{s^2 - \alpha_1\cdot\beta_1\cdot s - \alpha_0\cdot\alpha_1\cdot\beta_0} \longleftarrow \begin{cases} c = \alpha_0\cdot\alpha_1\cdot\gamma \\ d = -\alpha_0\cdot\beta_1 \\ e = -\alpha_0\cdot\alpha_1\cdot\beta_0 \end{cases}$$

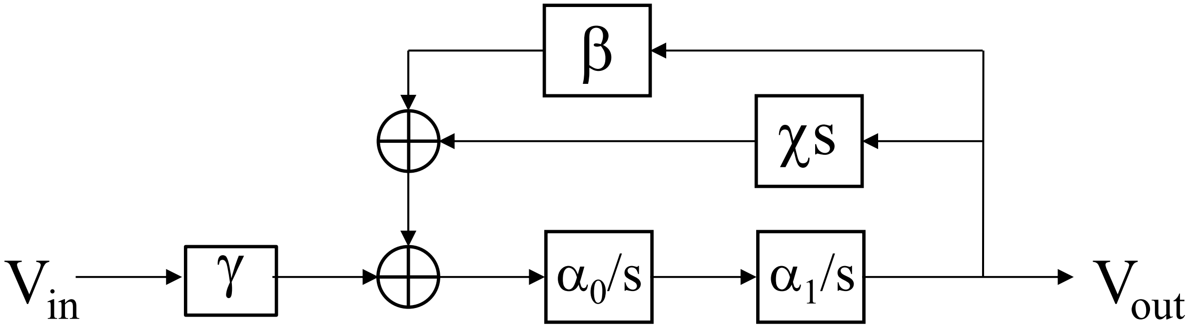
Block diagram of 2nd-order LPF 3

$$V_{out} = \frac{1}{s^2}(c\cdot V_{in} - d\cdot s\cdot V_{out} - e\cdot V_{out})$$

$$= \frac{\alpha_0}{s}\frac{\alpha_1}{s}\left(\frac{c}{\alpha_0\alpha_1}V_{in} - \frac{d}{\alpha_0\alpha_1}s\cdot V_{out} - \frac{e}{\alpha_0\alpha_1}V_{out}\right)$$

$$\equiv \frac{\alpha_0}{s}\frac{\alpha_1}{s}(\gamma\cdot V_{in} + \chi\cdot s\cdot V_{out} + \beta\cdot V_{out})$$

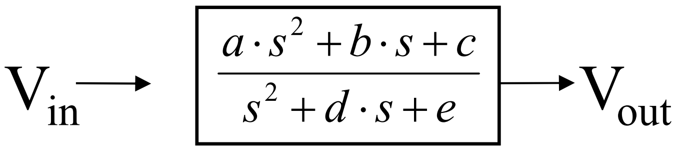




$$H(s) = \frac{V_{out}}{V_{in}} = \frac{c}{s^2+d\cdot s+e} = \frac{\alpha_0\cdot\alpha_1\cdot\gamma}{s^2-\alpha_0\cdot\alpha_1\cdot\chi\cdot s-\alpha_0\cdot\alpha_1\cdot\beta} \longleftarrow \begin{cases} c=\alpha_0\cdot\alpha_1\cdot\gamma \\ d=-\alpha_0\cdot\alpha_1\cdot\chi \\ e=-\alpha_0\cdot\alpha_1\cdot\beta \end{cases}$$

Block diagram of 2nd-order transfer function

$$\frac{V_{out}}{V_{in}} = \frac{a \cdot s^2 + b \cdot s + c}{s^2 + d \cdot s + e}$$

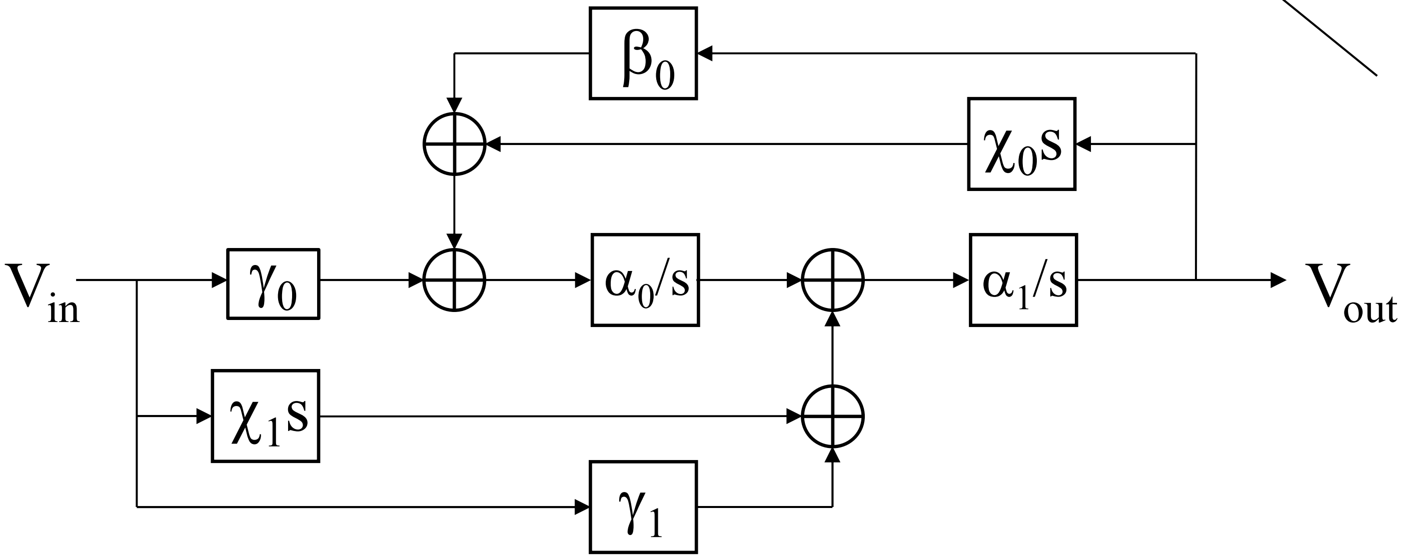


$$V_{out} = \frac{1}{s^2}(a \cdot s^2 \cdot V_{in} + b \cdot s \cdot V_{in} + c \cdot V_{in}) + \frac{1}{s^2}(-d \cdot s \cdot V_{out} - e \cdot V_{out})$$

$$= \frac{\alpha_0}{s}\frac{\alpha_1}{s}\frac{c}{\alpha_0 \alpha_1} \cdot V_{in} + \frac{\alpha_1}{s}(\frac{a}{\alpha_1} \cdot s \cdot V_{in} + \frac{b}{\alpha_1} \cdot V_{in}) + \frac{\alpha_0}{s}\frac{\alpha_1}{s}(-\frac{d}{\alpha_0 \alpha_1} \cdot s \cdot V_{out} - \frac{e}{\alpha_0 \alpha_1} \cdot V_{out})$$

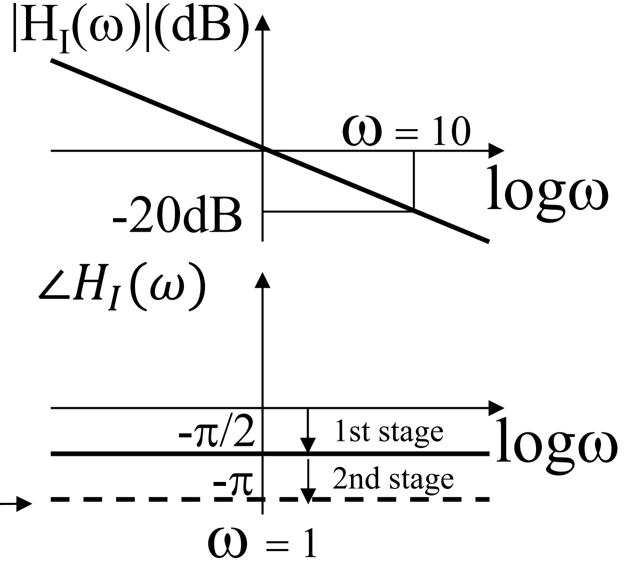
$$\equiv \frac{\alpha_0}{s}\frac{\alpha_1}{s}\gamma_0 \cdot V_{in} + \frac{\alpha_1}{s}(\chi_1 \cdot s \cdot V_{in} + \gamma_1 \cdot V_{in}) + \frac{\alpha_0}{s}\frac{\alpha_1}{s}(\chi_0 \cdot s \cdot V_{out} + \beta_0 \cdot V_{out})$$

$$\begin{cases} a = \alpha_1 \cdot \chi_1 \\ b = \alpha_1 \cdot \gamma_1 \\ c = \alpha_0 \cdot \alpha_1 \cdot \gamma_0 \\ d = -\alpha_0 \cdot \alpha_1 \cdot \chi_1 \\ e = -\alpha_0 \cdot \alpha_1 \cdot \beta_0 \end{cases}$$



Stability of higher order transfer function

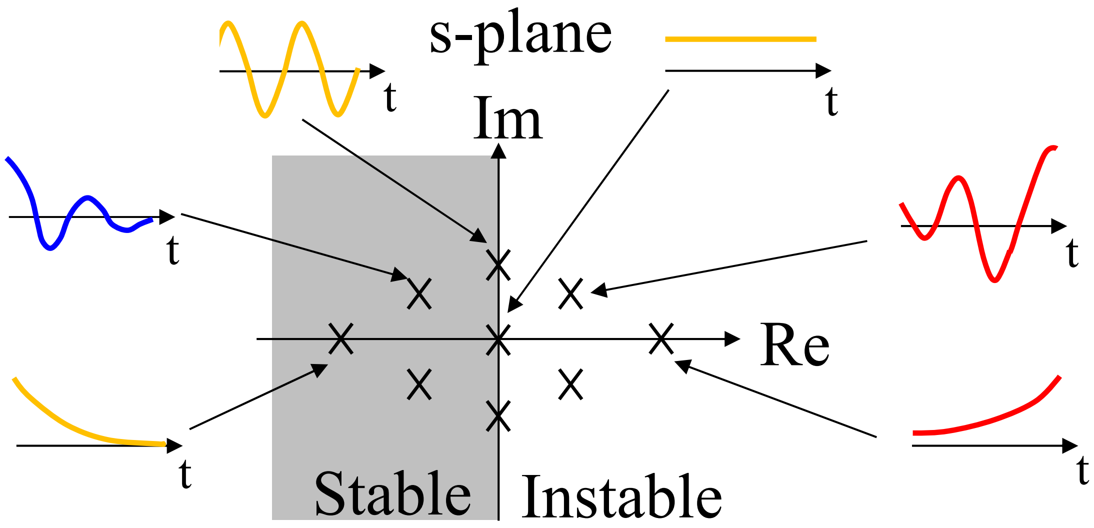
- A 2-stage-cascade connection of the integrators or differentiators can cause instability of the circuit with the feedback loop, because of the phase rotation > π. In the worst case, the circuit unexpectedly oscillates.



Bode diagram of integrator

The feedback loop with the phase shift > π causes the negative feedback (NFB) or positive feedback (PFB).

Stability of transfer functions



Impulse responses

You can simply cheek the stability of the circuit with the pole arrangement of the transfer function. The poles on the imaginary axis and right half plane cause the instability or the metastability of the circuit.

3.2 Block diagram of discrete-time circuit

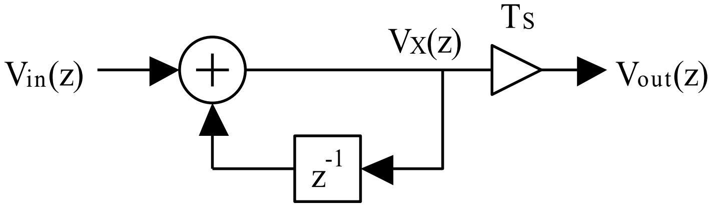
Integrator and differentiator (1)

Integrator by BET

$$V_{out}(z) = \frac{T_S}{1 - z^{-1}} V_{in}(z)$$

Differentiator by BET

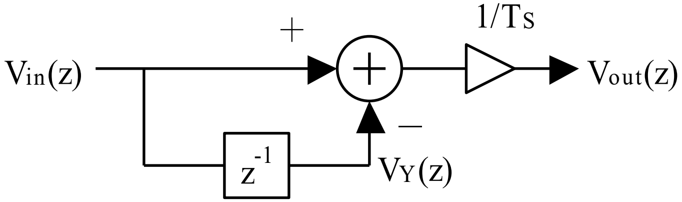
$$V_{out}(z) = \frac{1 - z^{-1}}{T_S} V_{in}(z)$$



$$V_X(z) = V_{in}(z) + z^{-1} V_X(z)$$

$$V_X(z) = \frac{V_{in}(z)}{1 - z^{-1}}$$

$$V_{out}(z) = T_S \cdot V_X(z) = \frac{T_S}{1 - z^{-1}} V_{in}(z)$$



$$V_Y(z) = z^{-1} V_{in}(z)$$

$$V_{out}(z) = \frac{1}{T_S} \{-z^{-1} V_{in}(z) + V_{in}(z)\}$$

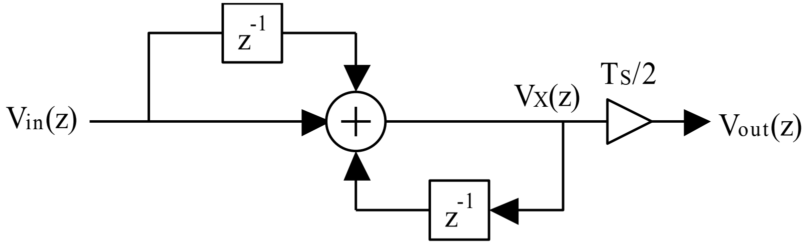
$$= \frac{1 - z^{-1}}{T_S} V_{in}(z)$$

Integrator and differentiator (2)

Integrator by bilinear transform	Differentiator by bilinear transform

$$V_{out}(z) = \frac{T_S}{2}\frac{1+z^{-1}}{1-z^{-1}}V_{in}(z)$$

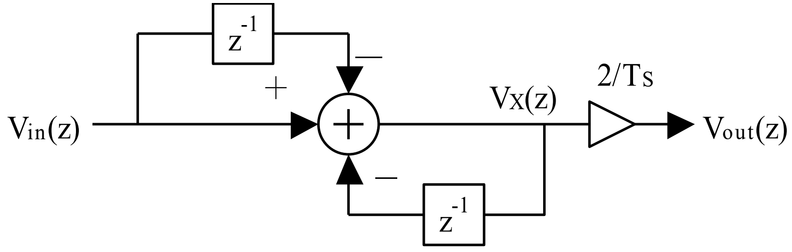
$$V_{out}(z) = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}V_{in}(z)$$



$$V_X(z) = (1+z^{-1})V_{in}(z) + z^{-1}V_X(z)$$

$$(1-z^{-1})V_X(z) = (1+z^{-1})V_{in}(z)$$

$$V_{out}(z) = \frac{T_S}{2}V_X(z) = \frac{T_S}{2}\frac{1+z^{-1}}{1-z^{-1}}V_{in}(z)$$



$$V_X(z) = (1-z^{-1})V_{in}(z) - z^{-1}V_X(z)$$

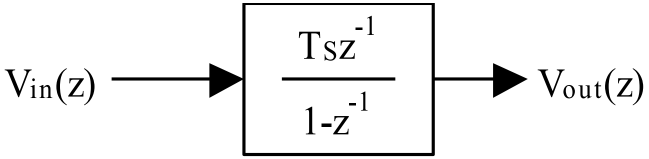
$$(1+z^{-1})V_X(z) = (1-z^{-1})V_{in}(z)$$

$$V_{out}(z) = \frac{2}{T_S}V_X(z) = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}V_{in}(z)$$

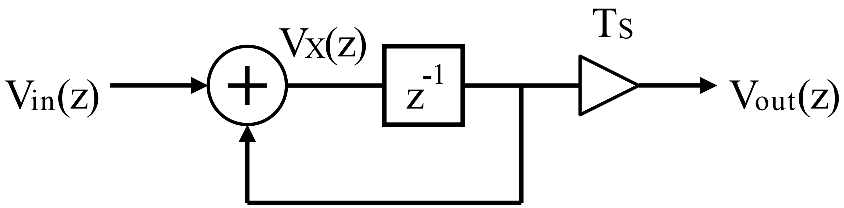
Integrator

Integrator by FET

$$V_{out}(z) = \frac{T_S \cdot z^{-1}}{1 - z^{-1}} V_{in}(z)$$



The FET integrator is equivalent to the BET integrator + the delay element of T_s.

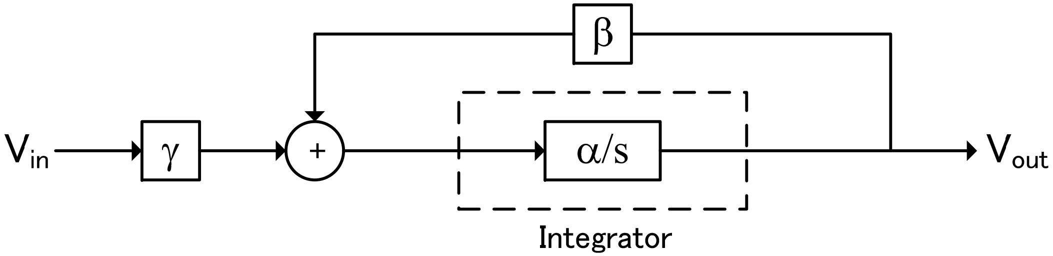


NOTE: The integrator and delaying integrator show the same characteristic except for delay time. The delaying integrator does not output the hazard generated by the adder.

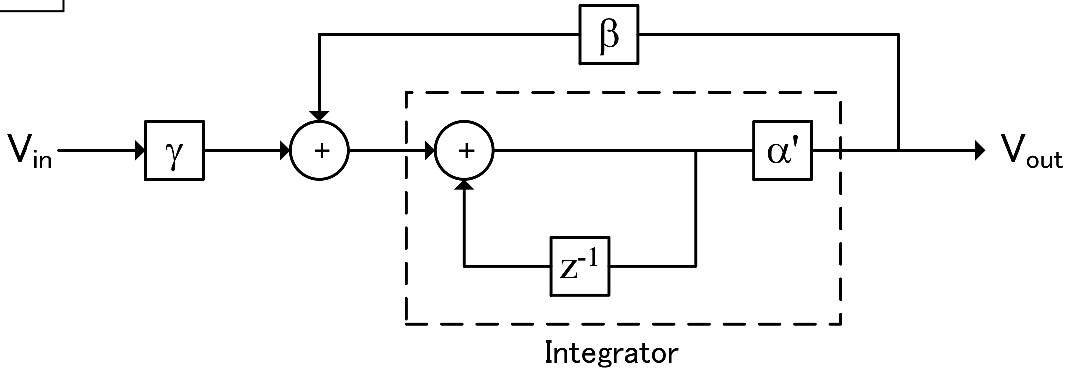
Block diagram of variable s and z

Examples of 1st-order LPF

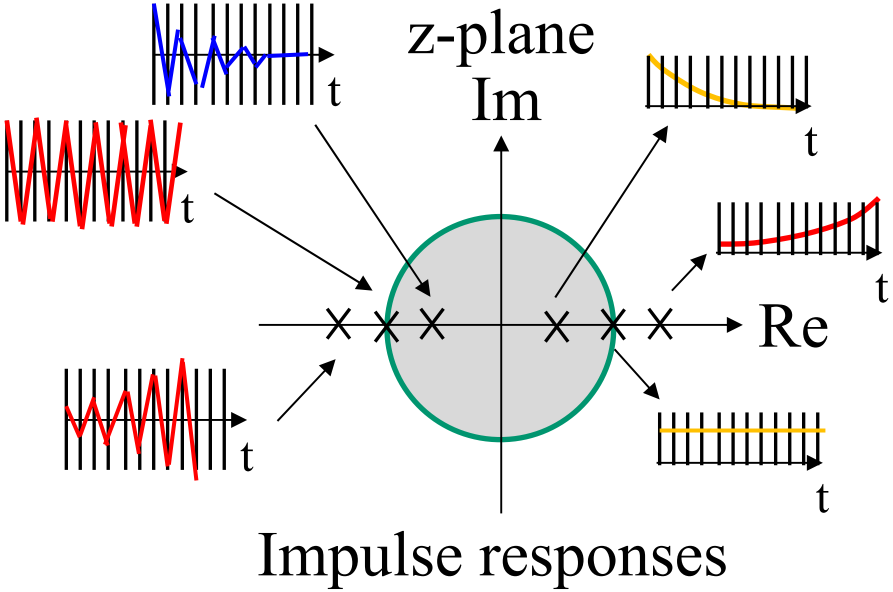
variable s Continuous-time analog circuits



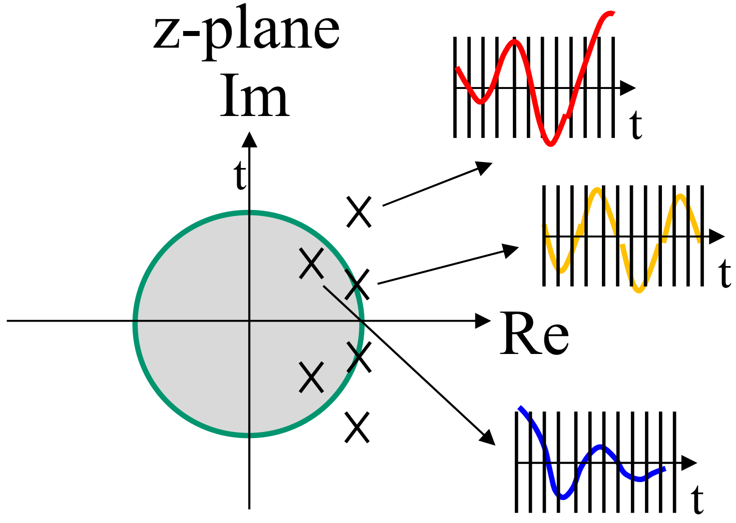
variable z Discrete-time analog circuits and digital circuits



Stability of transfer functions



Impulse responses



Impulse responses

You can simply cheek the stability of the circuit with the pole arrangement of the transfer function. The poles on the circumference or in the exterior of the unit circle cause the instability or the metastability of the circuit.

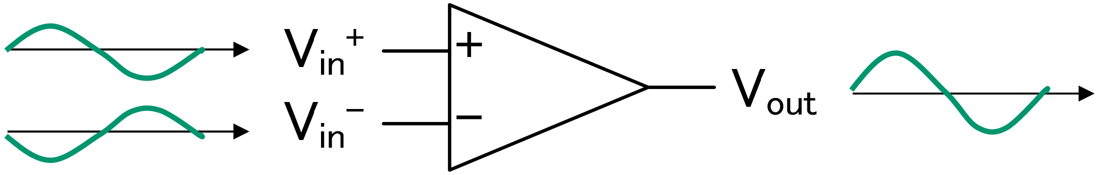
3.3 Continuous-time analog implementation

Implementation methods of analog circuits

Signal	**Circuitry**	**Components**	**Features**
Continuous time	RC	R, C, OPA	High precision, low power consumption
	gm-C	C, OTA	High speed, low power consumption
Discrete time	Switched capacitor	C, CMOS-switch, OPA	Very high precision
	Switched Current	Current mirror, CMOS-switch	Very high speed

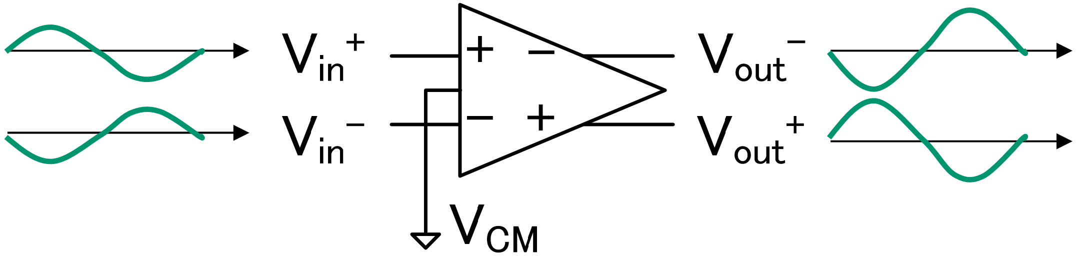
2 types of OPA

Function



$$\begin{cases} V_{out} = A_d(V_{in}^+ - V_{in}^-) \\ \text{A}_\text{d} = \text{Differential Gain} \end{cases}$$

Symbol of Single-end OPA

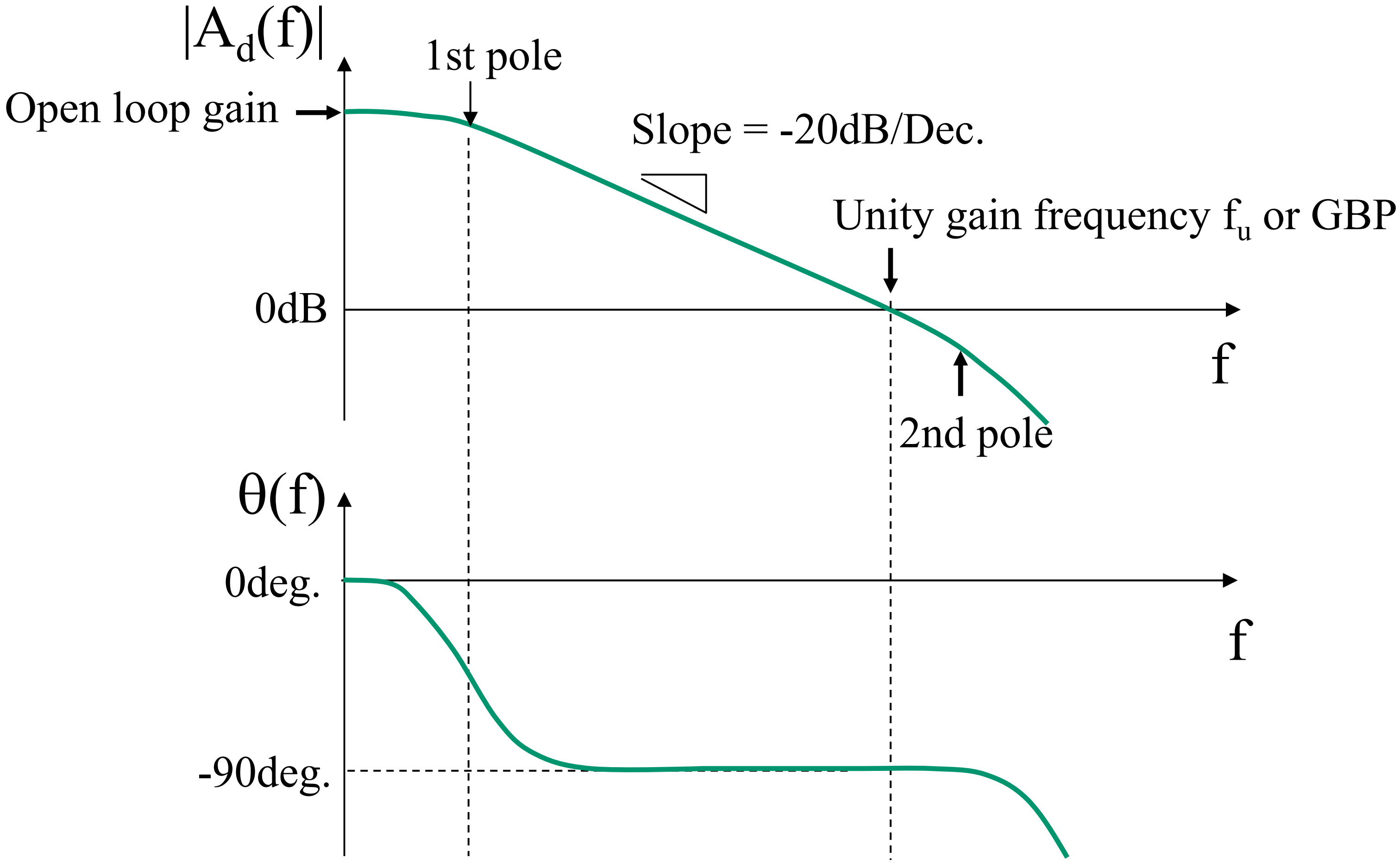


$$\begin{cases} V_{out}^+ = \dfrac{A_d}{2}(V_{in}^+ - V_{in}^-) \\ V_{out}^- = -\dfrac{A_d}{2}(V_{in}^+ - V_{in}^-) \\ V_{out} = V_{out}^+ - V_{out}^- \end{cases}$$

Symbol of Full-differential OPA

A_d = Differential Gain

AC characteristic of OPA



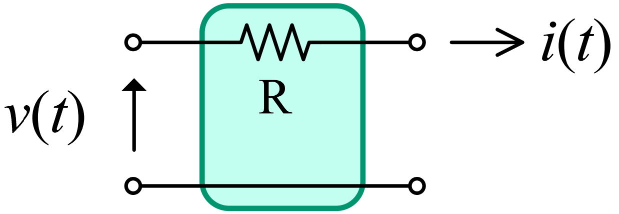
Linear operators

Linear operation elements in the analog circuits

Operation	Symbol	(Typical implementation)
Addition and subtraction	$\mathrm{In}_1 \xrightarrow{\pm} \overset{\mathrm{In}_2 \downarrow \pm}{\oplus} \longrightarrow \mathrm{Out}$	(Current summing)
Constant factor	$\mathrm{In} \longrightarrow \boxed{b} \longrightarrow \mathrm{Out}$	(Resistor)
Derivation	$\mathrm{In} \longrightarrow \boxed{cs} \longrightarrow \mathrm{Out}$	(Capacitor)
Integral	$\mathrm{In} \longrightarrow \boxed{a/s} \longrightarrow \mathrm{Out}$	(OPA or OTA)

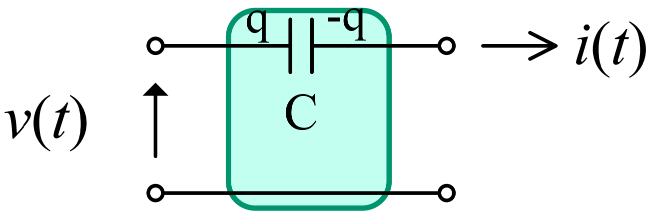
a, b, c : circuit constants

Continuous-time analog operators



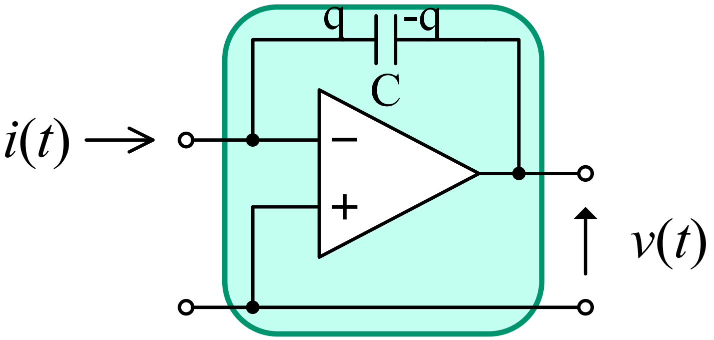
Constant multiplication

$$i(t)=\frac{1}{R}v(t)=Gv(t)\overset{\mathcal{L}}{\Rightarrow}I(s)=G\cdot V(s)$$



Voltage differentiation

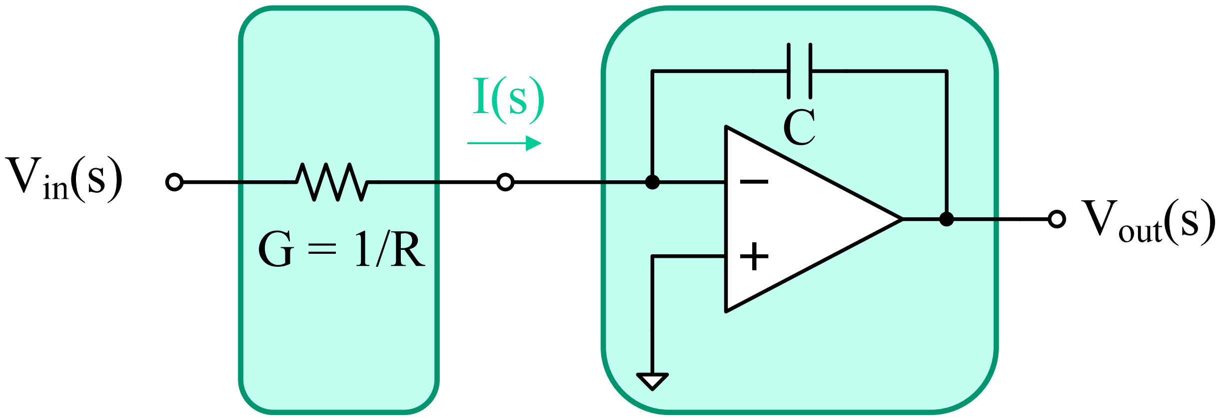
$$i(t)=\frac{dq(t)}{dt}=C\frac{dv(t)}{dt}\overset{\mathcal{L}}{\Rightarrow}I(s)=sC\cdot V(s)$$

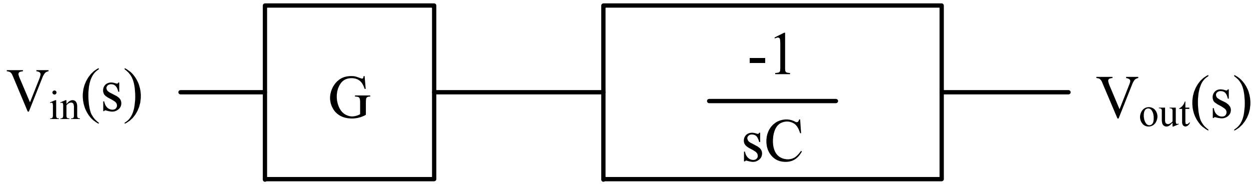


Current integration

$$v(t)=-\frac{1}{C}q(t)=-\frac{1}{C}\int_0^t i(\tau)d\tau\overset{\mathcal{L}}{\Rightarrow}V(s)=\frac{-1}{sC}I(s)$$

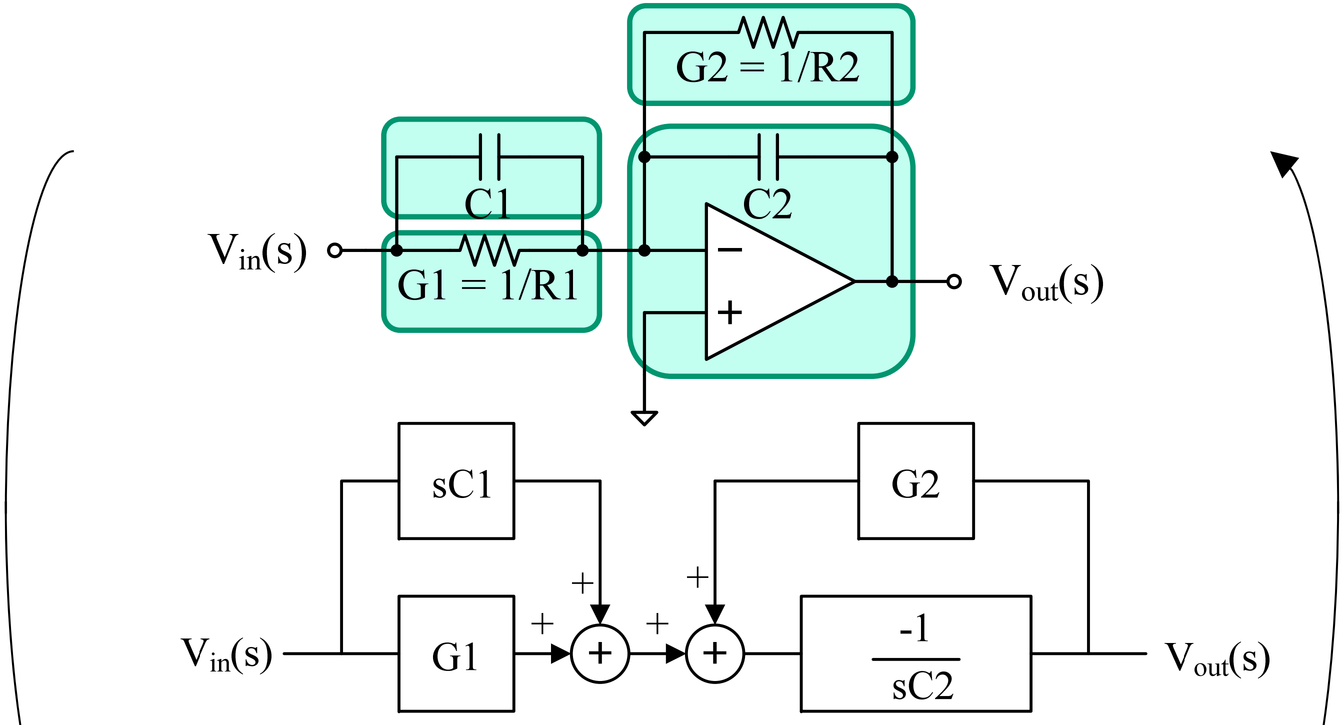
Continuous-time analog integrator (CAI)





Transfer function of CAI: $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-\frac{G}{C}}{s}$

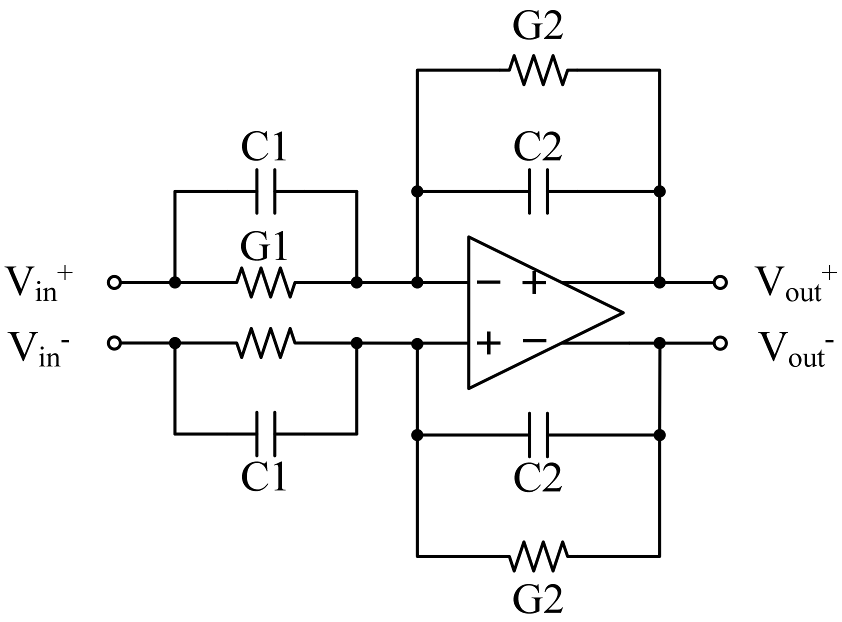
Implementation example 1

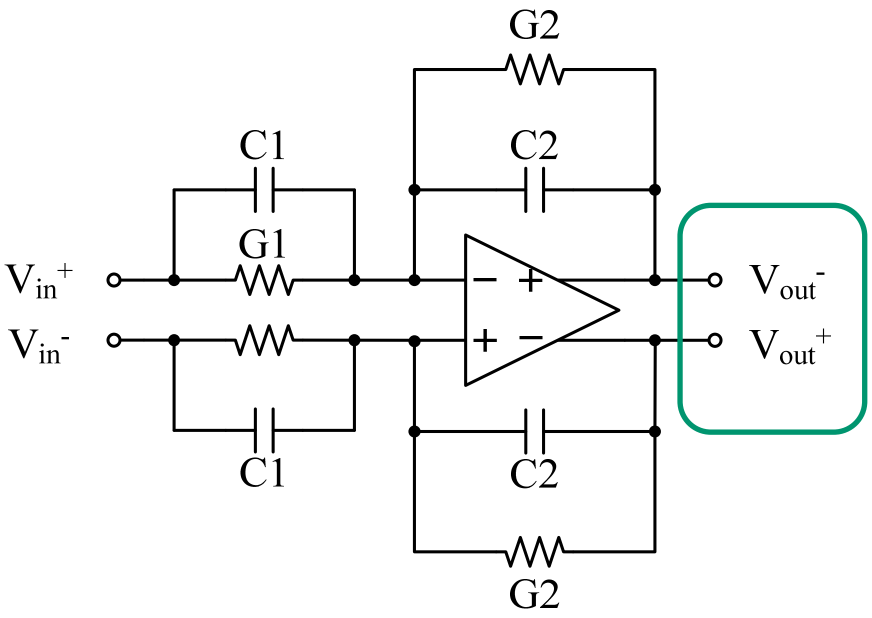


$$V_{out} = \frac{-1}{sC2}\left(sC1 \cdot V_{in} + \frac{1}{R1}V_{in} + \frac{1}{R2}V_{out}\right)$$

$$H(s) = \frac{V_{out}}{V_{in}} = -\frac{C1}{C2}\frac{s + \frac{G1}{C1}}{s + \frac{G2}{C2}} = a\frac{s + b}{s + c}$$

Full-differential implementation

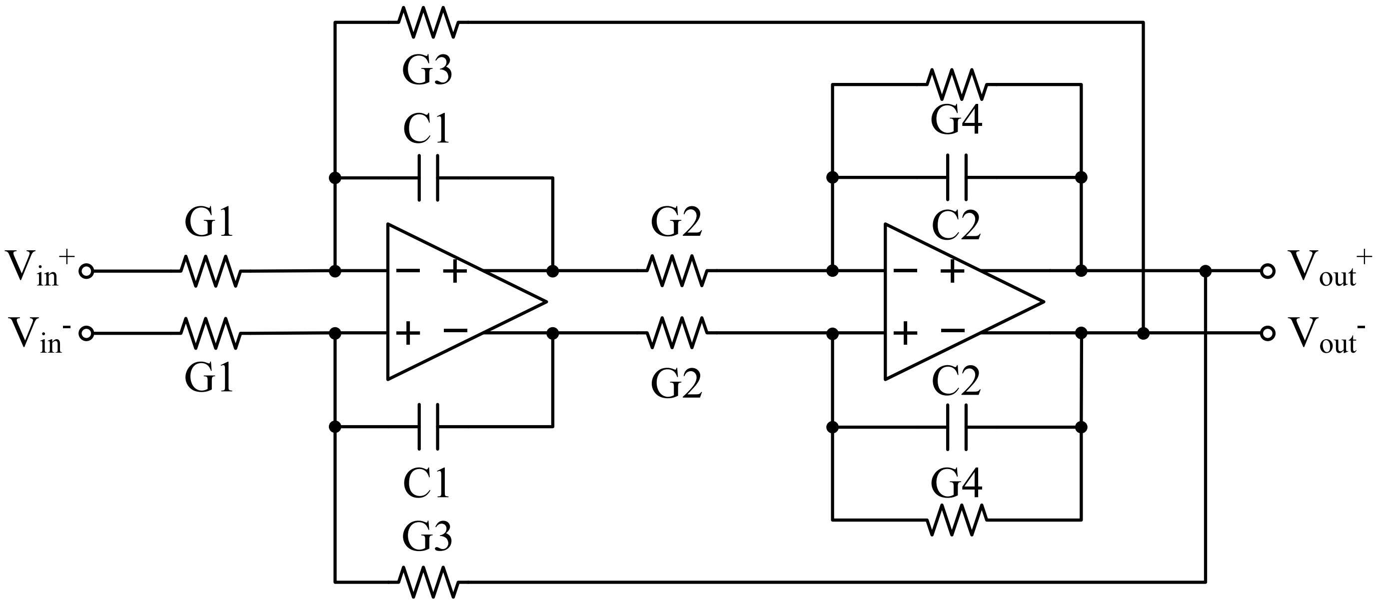


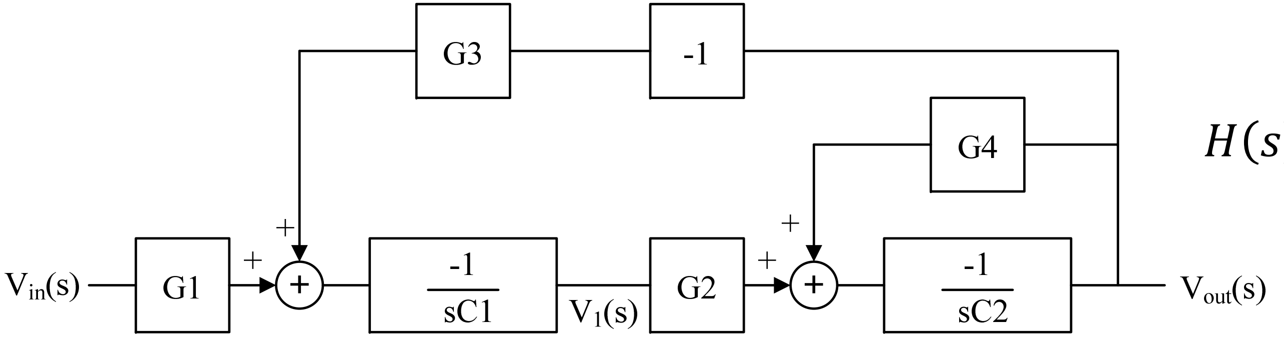


$$H(s) = \frac{V_{out}}{V_{in}} = -\frac{C1}{C2}\frac{s+\frac{G1}{C1}}{s+\frac{G2}{C2}}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{C1}{C2}\frac{s+\frac{G1}{C1}}{s+\frac{G2}{C2}}$$

Implementation example 2





$$H(s) = \frac{\frac{G1G2}{C1C2}}{s^2 + \frac{G4}{C2}s + \frac{G2G3}{C1C2}}$$

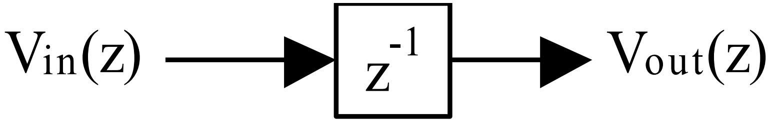
3.4 Discrete-time analog implementation

Delay element

$$x_d(t - T_S) = x(t) \cdot \delta(t - T_S)$$

$$\int_0^\infty x_d(t - T_S) \cdot e^{-st} dt = \int_0^\infty x(t) \cdot \delta(t - T_S) \cdot e^{-st} dt$$

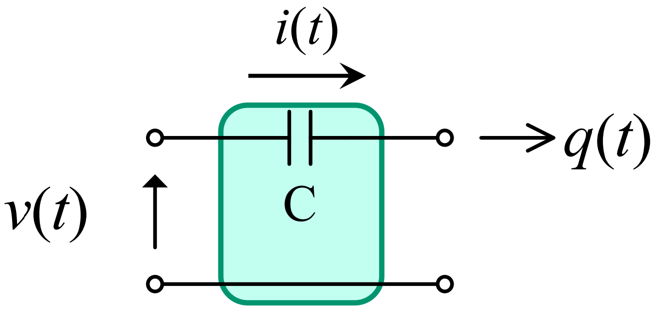
$$= x(T_S) e^{-sT_S} = x(T_S) \cdot z^{-1}$$



$$V_{in}(z) \longrightarrow \boxed{z^{-1}} \longrightarrow V_{out}(z)$$

$v(t)$ → Time shift for 1-cycle → $v(t - T_S)$

Discrete-time analog operators 1

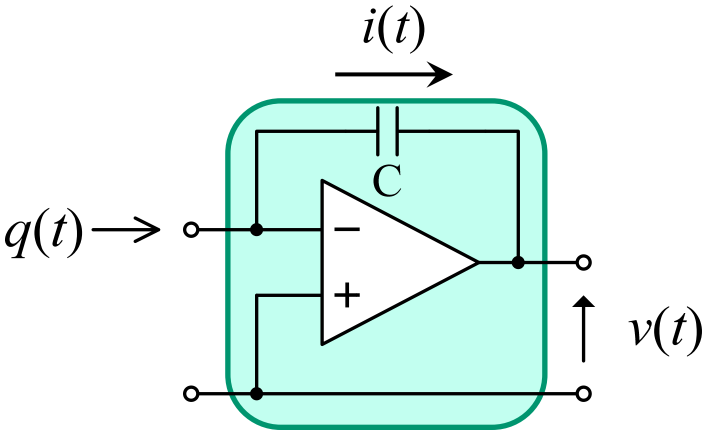


Voltage derivation

$$I(s) = sC \cdot V(s) \overset{BET}{\Longrightarrow} I(z) = C(1 - z^{-1})\frac{1}{T_S}\mathrm{V(z)}$$

$$Q(z) = I(z)T_S = C(1 - z^{-1})$$

$$\mathrm{V(z)} \;\text{—}\; \boxed{\mathrm{C}\,(1 - \mathrm{z}^{-1})} \;\text{—}\; \mathrm{Q(z)}$$



Charge integration

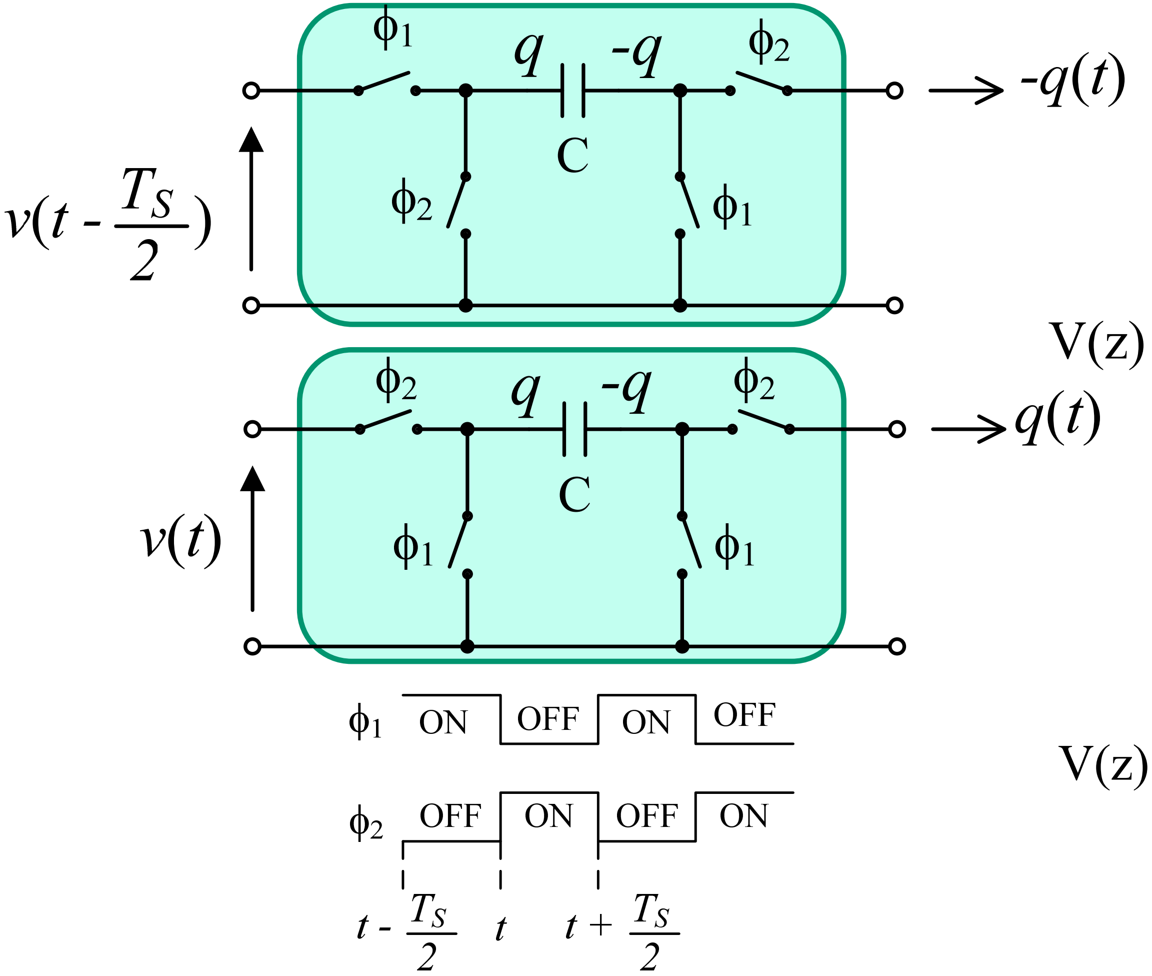
$$V(s) = \frac{-1}{sC} I(s) \overset{FET}{\Longrightarrow} V(z) = \frac{-1}{C}\frac{1}{1 - z^{-1}} I(z)T_S$$

$$V(z) = \frac{-1}{C}\frac{1}{1 - z^{-1}} Q(z)$$

$$\mathrm{Q(z)} \;\text{—}\; \boxed{\frac{-1}{\mathrm{C}\,(1 - \mathrm{z}^{-1})}} \;\text{—}\; \mathrm{V(z)}$$

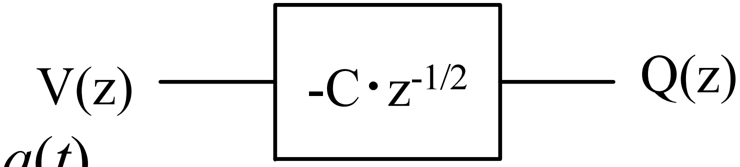
Discrete-time analog operators 2

Constant multiplication (Switched capacitor)



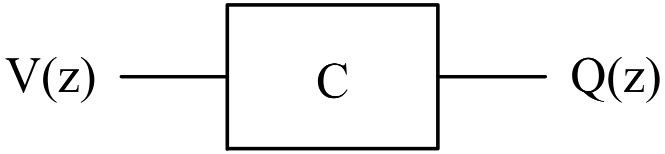
$$-q(t) = C \cdot v\left(t - \frac{T_S}{2}\right)$$

$$\overset{Z}{\Rightarrow} -Q(z) = C \cdot V(z) z^{-\frac{1}{2}}$$

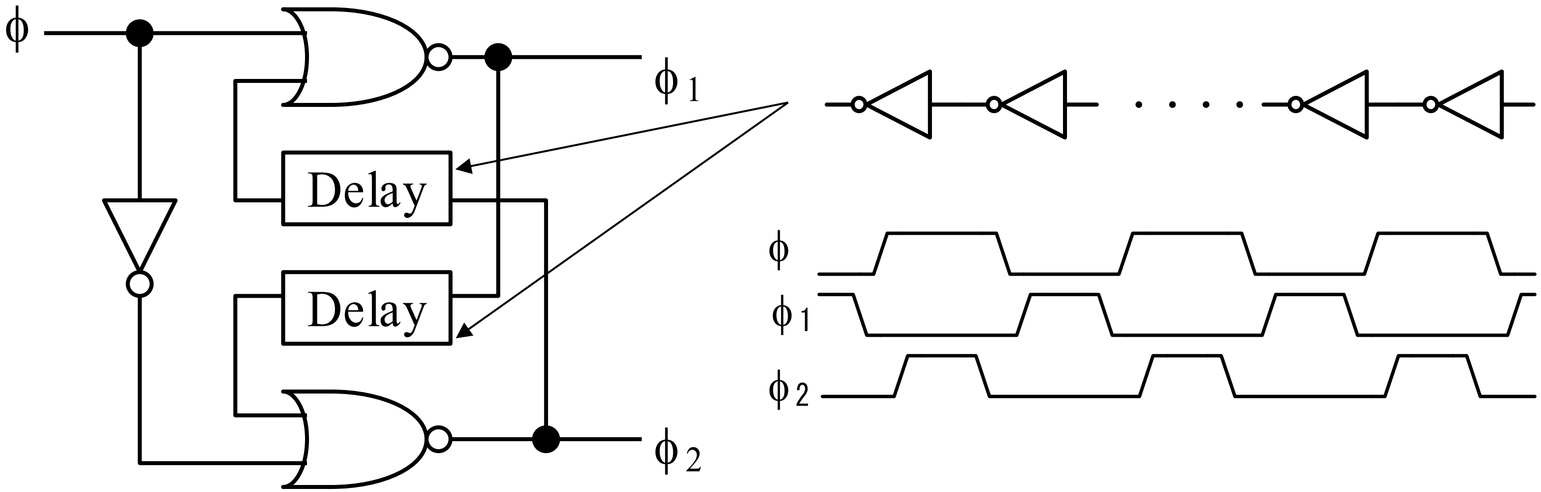


$$q(t) = C \cdot v(t)$$

$$\overset{Z}{\Rightarrow} Q(z) = C \cdot V(z)$$



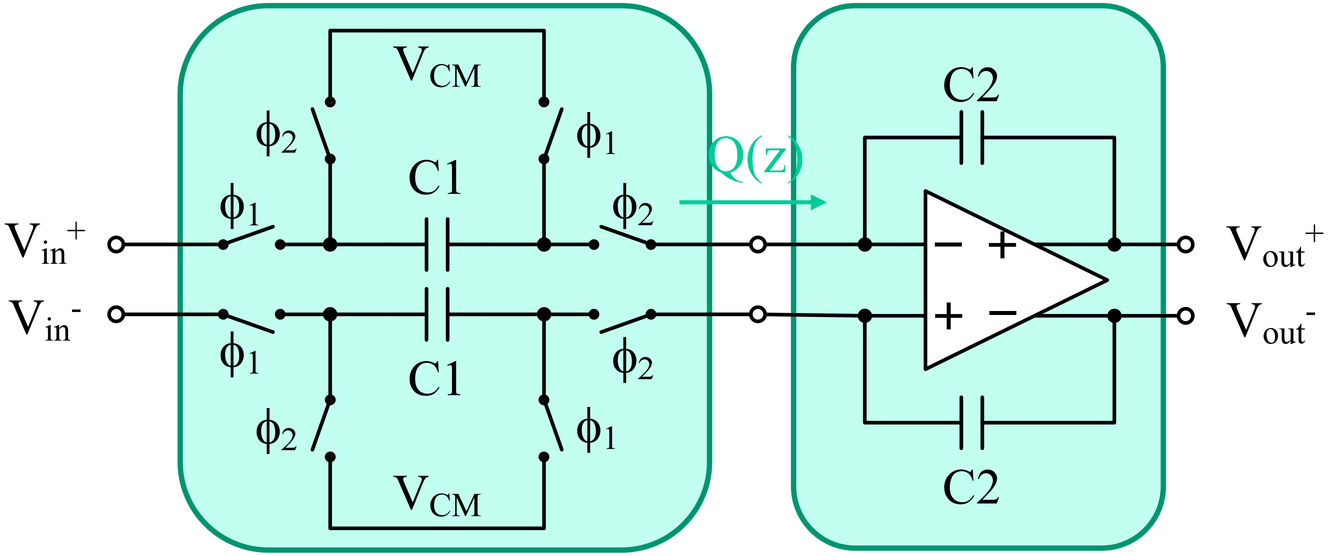
Non-overlapping clock generator

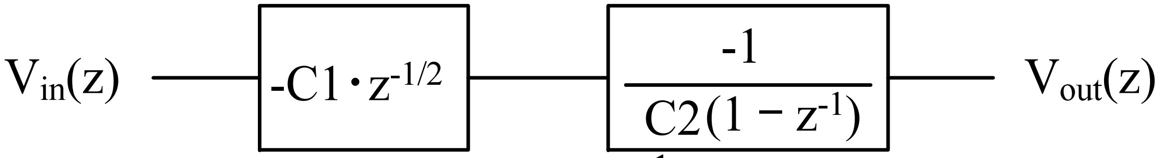


The two-phase clocks ϕ_1 and ϕ_2 are required to drive the switched capacitor. The edge of clocks ϕ_1 and ϕ_2 must not overlap, because the leak of the charge stored in capacitors causes the error of the signal processing.

DAI: discrete-time analog integrator 1

Positive phase integrator

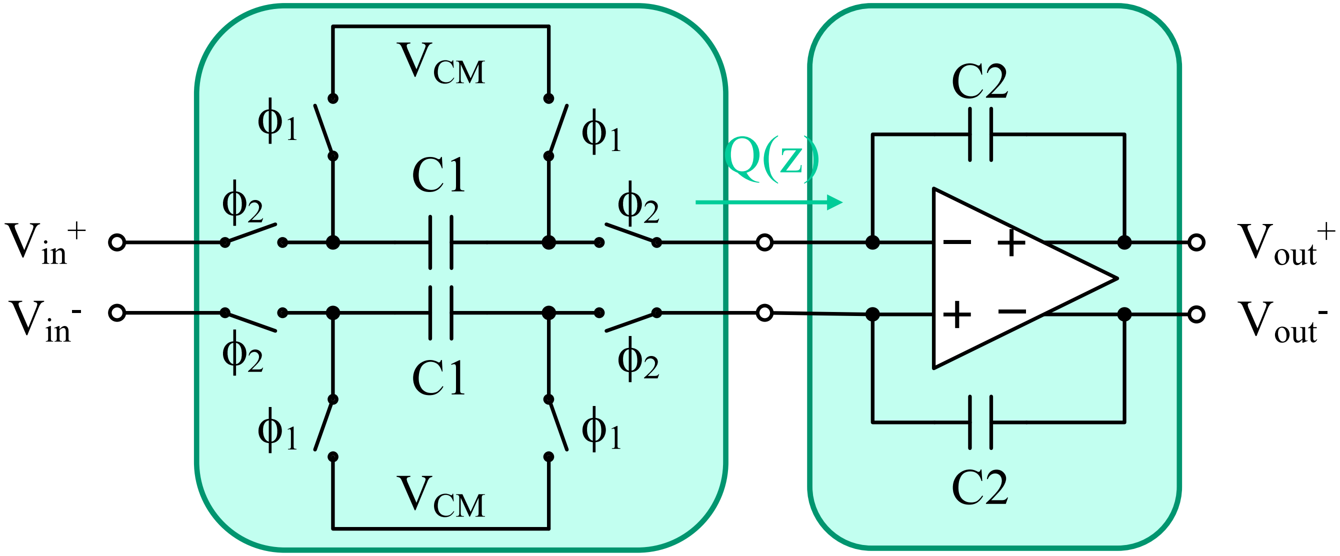


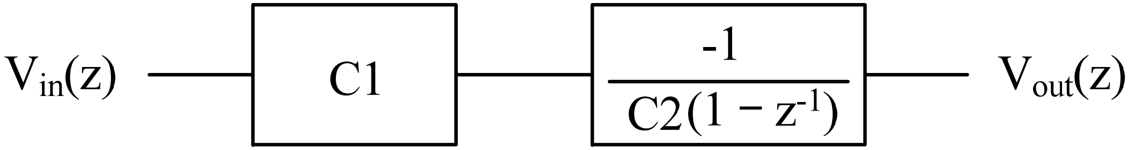


$$H(z) = \frac{C1}{C2}\frac{z^{-\frac{1}{2}}}{1-z^{-1}}$$

DAI: discrete-time analog integrator 2

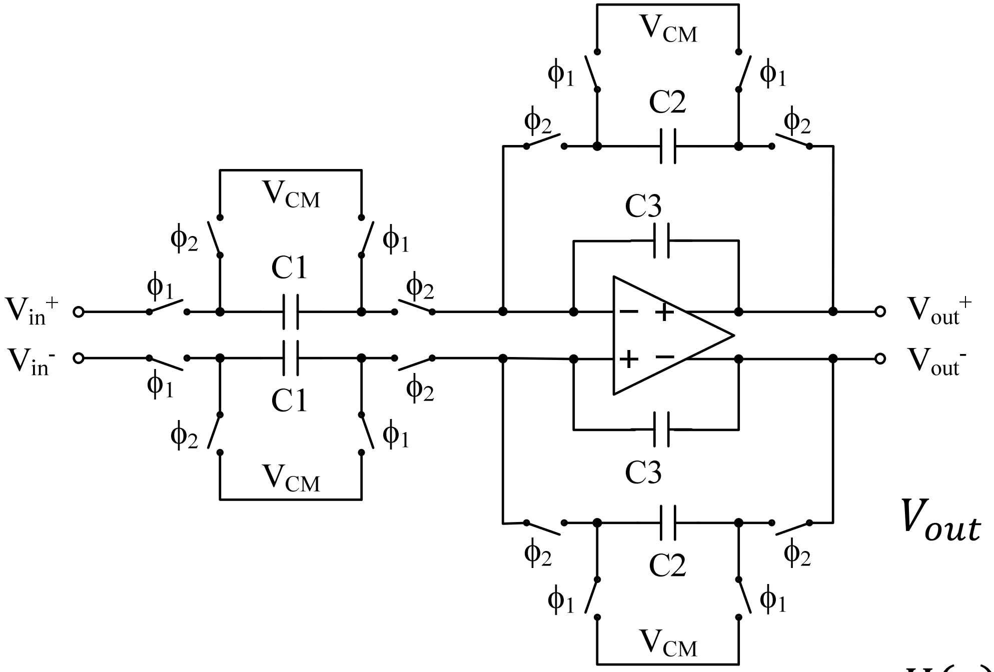
Opposite phase integrator





$$H(z) = -\frac{C1}{C2}\frac{1}{1-z^{-1}}$$

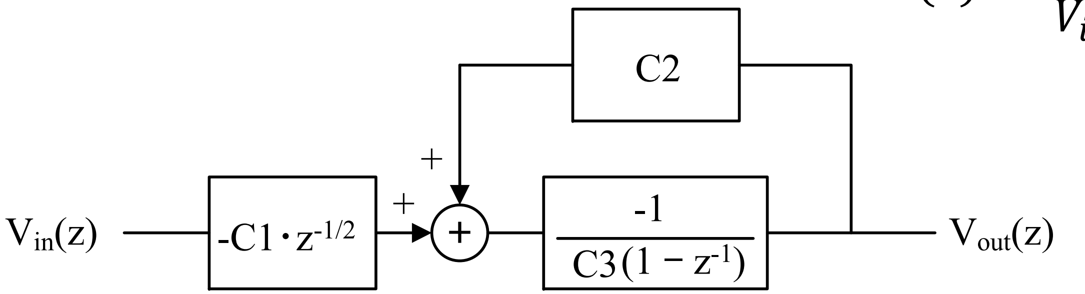
Implementation example



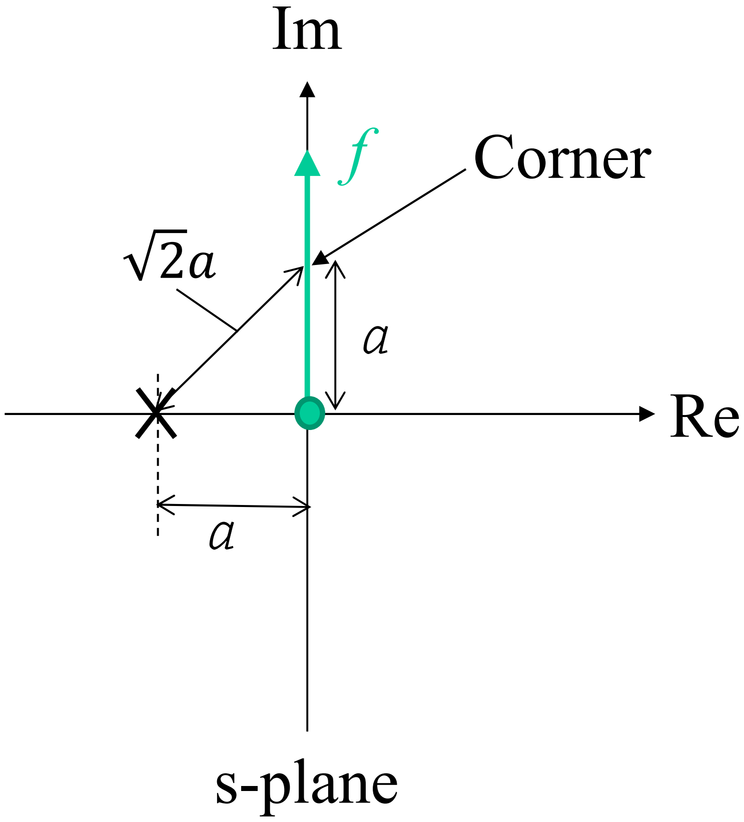
$$V_{out} = -\frac{1}{C3}\frac{1}{1-z^{-1}}\left(-C1z^{-\frac{1}{2}}V_{in} + C2 \cdot V_{out}\right)$$

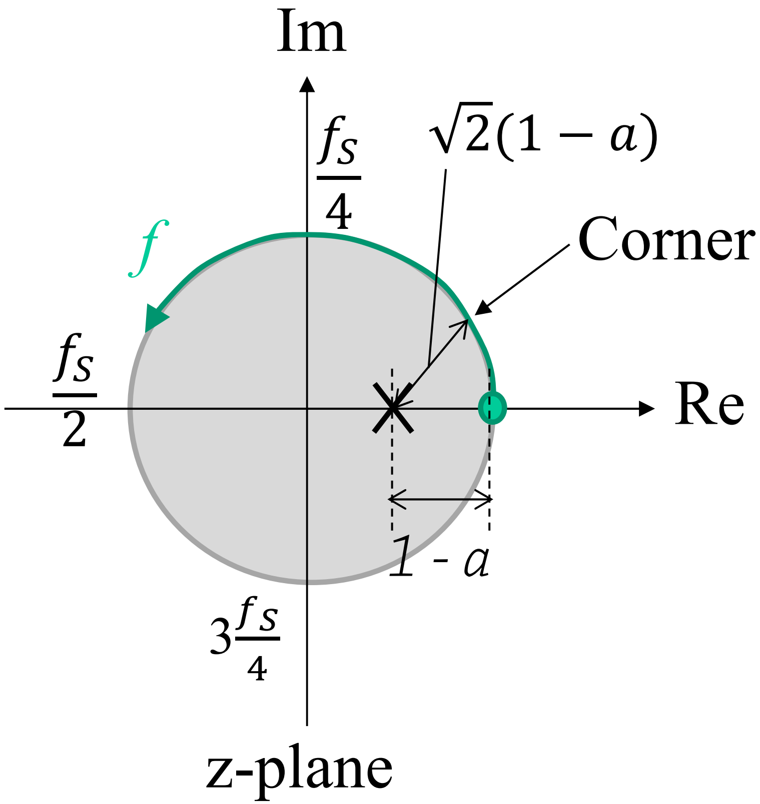
$$H(z) = \frac{V_{out}(z)}{V_{in}(z)} = -\frac{C1}{C2+C3}\frac{z^{-\frac{1}{2}}}{z-\frac{C3}{C2+C3}}$$

(LPF)



Poles in s-plane and z-plane

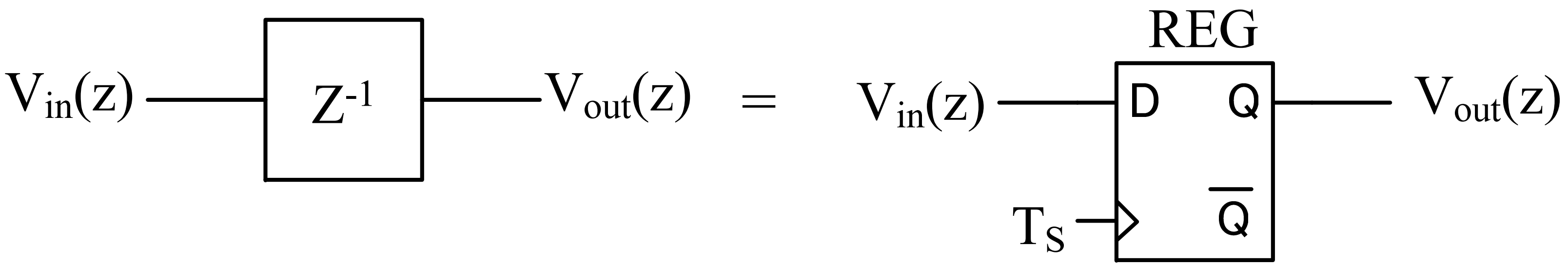




3.5 Digital implementation

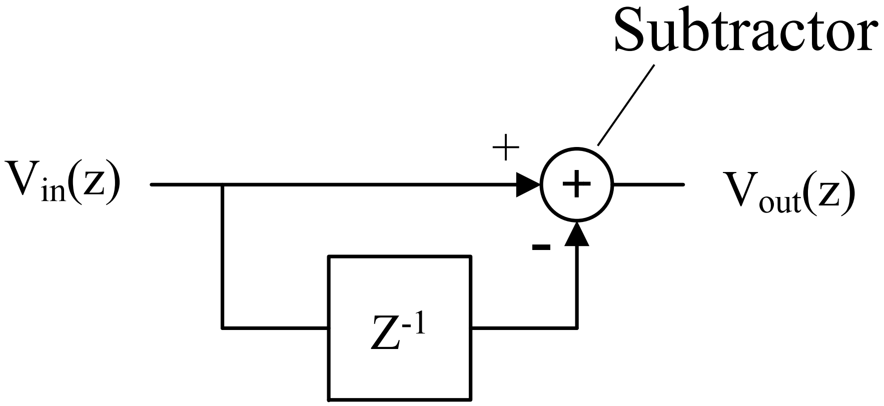
Digital operators 1

Delay



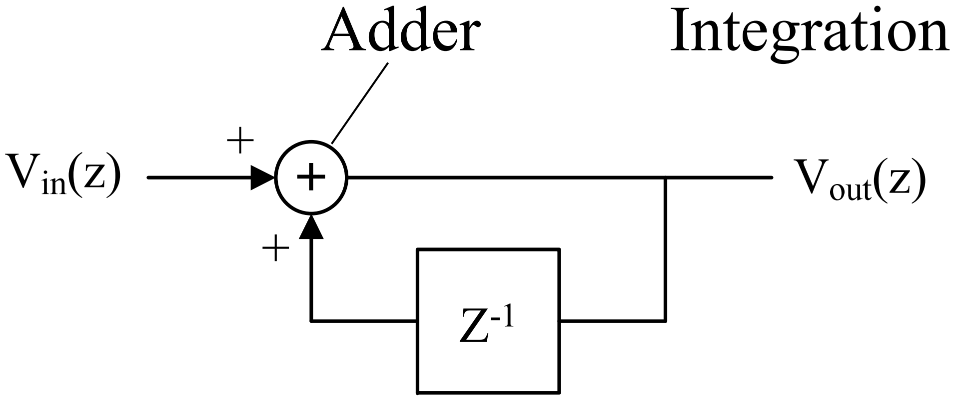
Differentiation (BET)

$$V_{out}(z) = (1 - z^{-1})V_{in}(z)$$



Digital operators 2

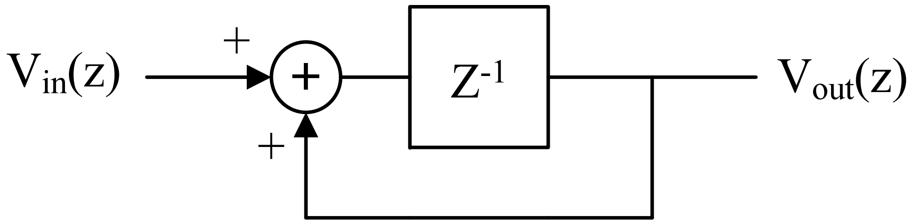
Integration (BET)



$$V_{out}(z) = V_{in}(z) + z^{-1}V_{out}(z)$$

$$V_{out}(z) = \frac{1}{1 - z^{-1}}V_{in}(z)$$

Integration (FET)

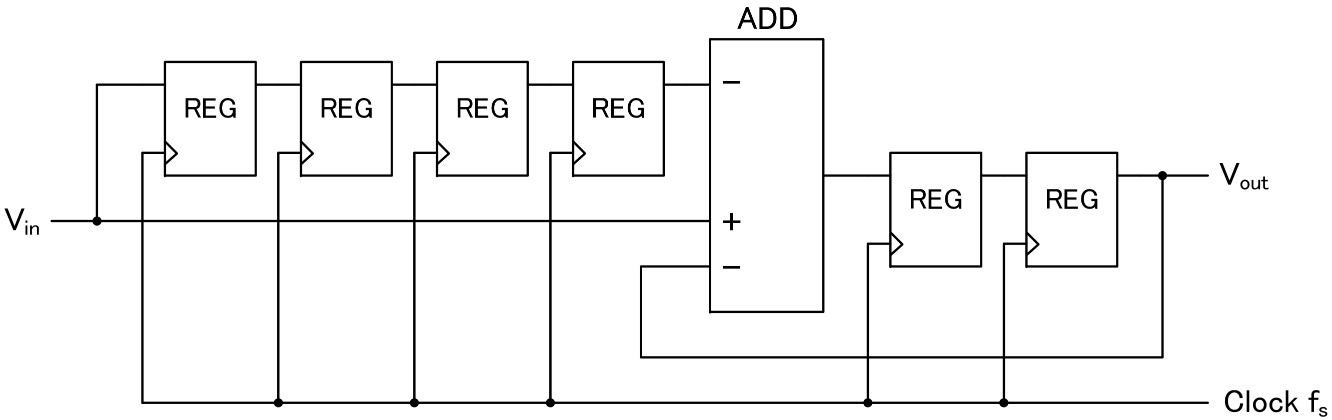


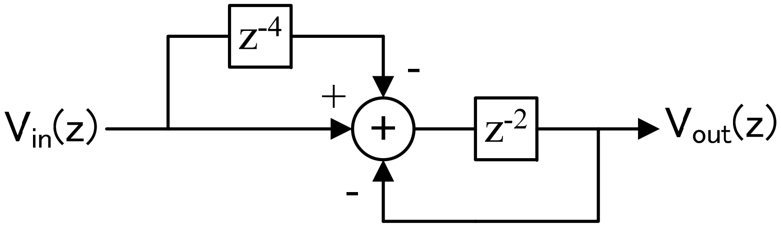
$$V_{out}(z) = z^{-1}(V_{in}(z) + V_{out}(z))$$

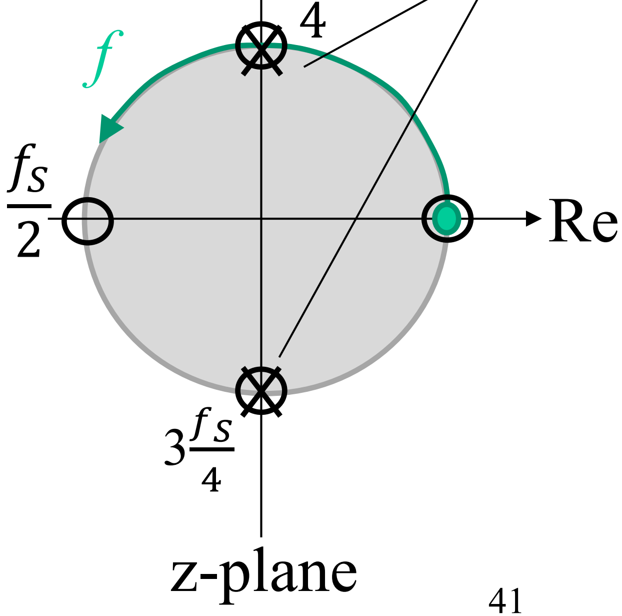
$$V_{out}(z) = \frac{z^{-1}}{1 - z^{-1}}V_{in}(z)$$

NOTE: The FET integrator does not output the hazards, because the digital delay element is the same as an register.

Implementation example







$$V_{out}(z) = z^{-2}(1 - z^{-4})V_{in}(z) - z^{-2}V_{out}(z)$$

$$= \frac{z^{-2}(1 - z^{-4})}{1 + z^{-2}} V_{in}(z)$$

Comb filter

Resonator