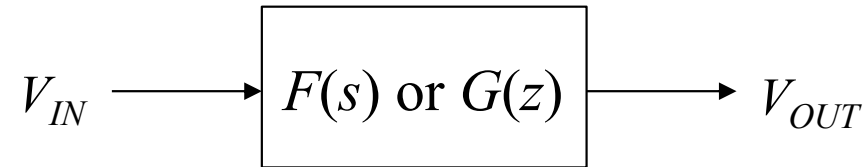


# 2. Transfer function

Kanazawa University  
Microelectronics Research Lab.  
Akio Kitagawa

# 2.1 Definition of transfer function

# Definition of transfer function



- Transfer function of CT circuit

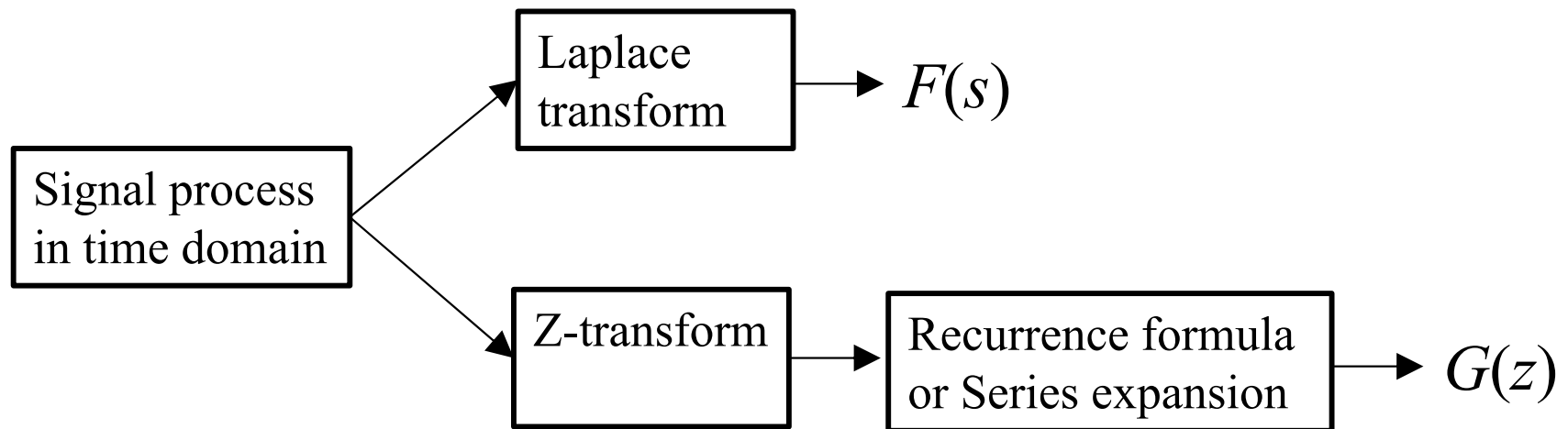
$$F(s) = \frac{V_{OUT}(s)}{V_{IN}(s)}$$

- Transfer function of DT circuit

$$G(z) = \frac{V_{OUT}(z)}{V_{IN}(z)}$$

[NOTE] A transfer function is defined in s-plane or z-plane.

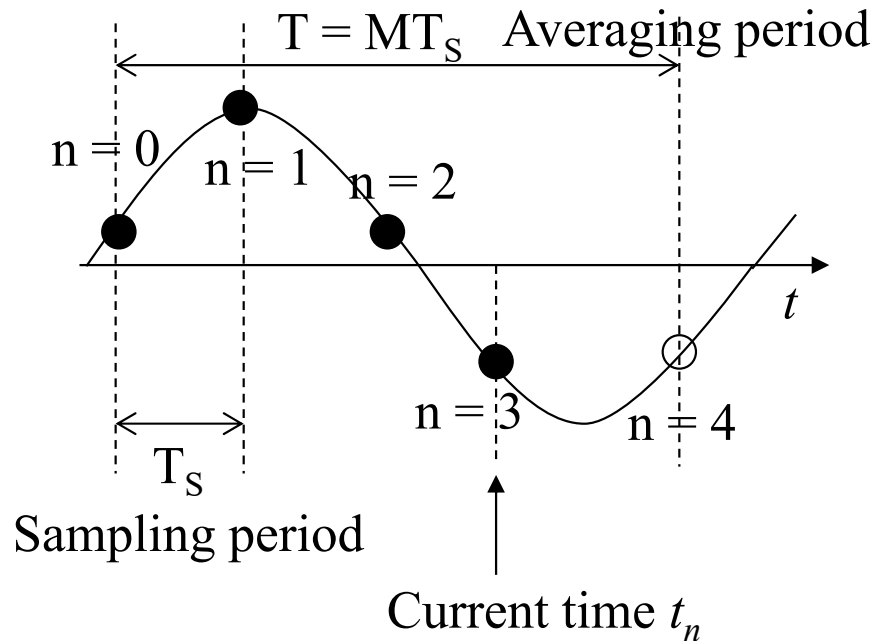
# Method for deriving transfer function



# Transfer function in z-domain

Transfer function of moving average

In the case of  $M = 4$



Signal process in time domain

$$y(t_n) = \frac{1}{T} \sum_{n=0}^{M-1} x(t_n) T_s = \frac{1}{M} \sum_{n=0}^{M-1} x(t_n)$$

$$= \frac{1}{M} \{x(t_0) + x(t_1) + x(t_2) + x(t_3)\}$$

$\downarrow \mathcal{Z}$

Series expansion of z variable

$$Y(z) = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} X(z)$$

$$= \frac{1}{M} (1 + z^{-1} + \dots + z^{-(M-1)}) X(z)$$

$$= \frac{1}{M} (z^{-3} + z^{-2} + z^{-1} + z^0) X(z)$$

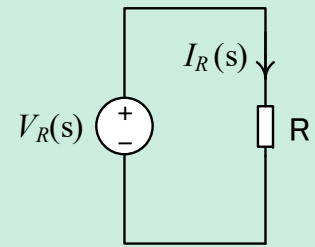
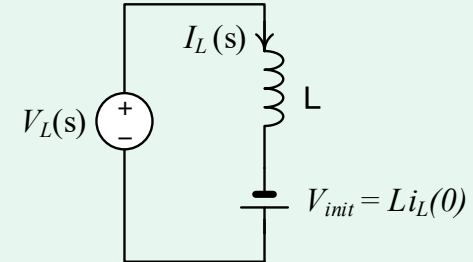
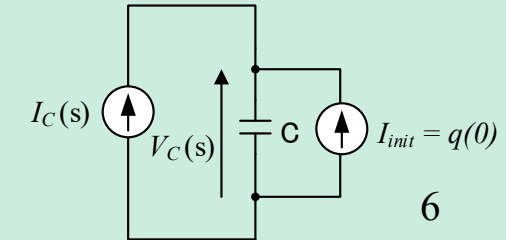
$\downarrow$

$$MY(z) = (z^{-1} + z^{-2} + \dots + z^{-M}) X(z) + (z^0 - z^{-M}) X(z) = z^{-1} MY(z) + (1 - z^{-M}) X(z)$$

$$(1 - z^{-1}) MY(z) = (1 - z^{-M}) X(z) \longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M} \frac{(1 - z^{-M})}{(1 - z^{-1})}$$

# Note: Current–voltage characteristic

Circuit equations are derived from Kirchhoff's law and current-voltage (I-V) characteristic for each device. Therefore, it will be useful if you know about Laplace transformed current-voltage characteristics in advance.

Device	I-V characteristic in t-domain	I-V characteristics in s-domain	Equivalent circuits
R	$v_R(t) = R \cdot i_R(t)$	$V_R(s) = R \cdot I_R(s)$	
L	$v_L(t) = L \frac{di_L(t)}{dt}$	$V_L(s) + L \cdot i_L(0) = sL \cdot I_L(s)$	
C	$v_C(t) = \frac{1}{C} \int_0^t i_C(\tau) d\tau$	$V_C(s) = \frac{1}{sC} \{I_C(s) + q(0)\}$	

# Transfer function in s-domain

$$\begin{cases} v_{in}(t) = \frac{1}{C} \int_0^{\infty} i_{in}(t) dt + Ri_{in}(t) \\ v_{out}(t) = Ri_{in}(t) \end{cases}$$

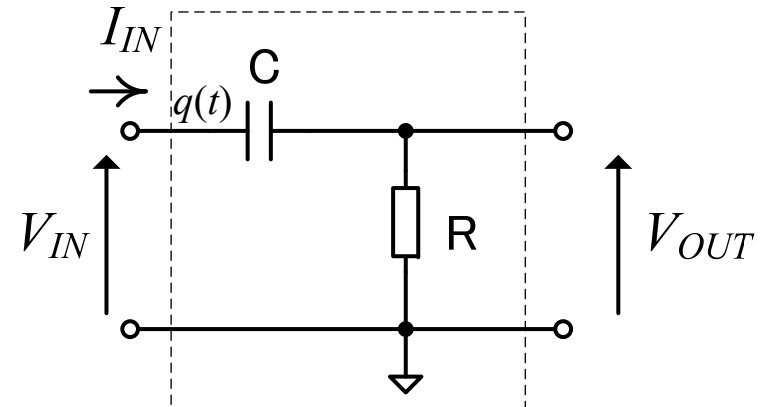
$\mathcal{L}$  (Initial condition:  $q(0)$ )

$$\begin{cases} V_{IN}(s) = \frac{1}{C} \left( \frac{I_{IN}(s)}{s} + \frac{q(0)}{s} \right) + RI_{IN}(s) \\ V_{OUT}(s) = RI_{IN}(s) \end{cases}$$

$$V_{OUT}(s) = \frac{R}{\frac{1}{sC} + R} V_{IN}(s) - \frac{R}{1 + sCR} q(0)$$

Transfer function for input signal

Transfer function for initial condition



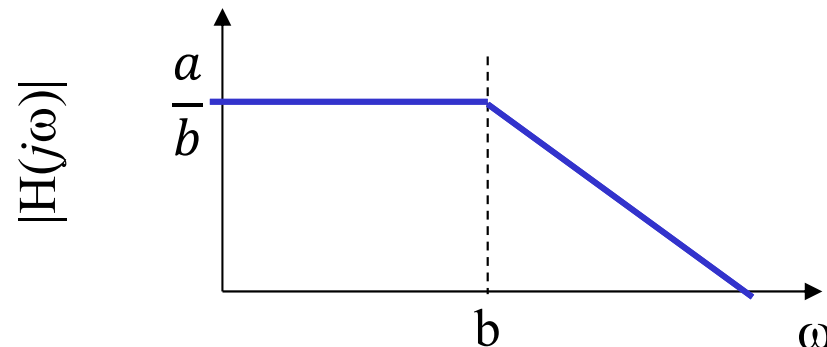
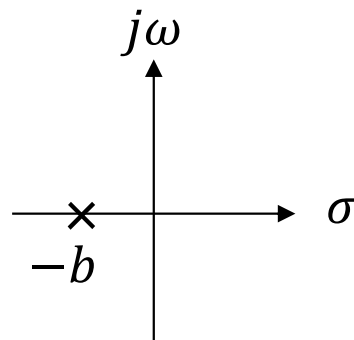




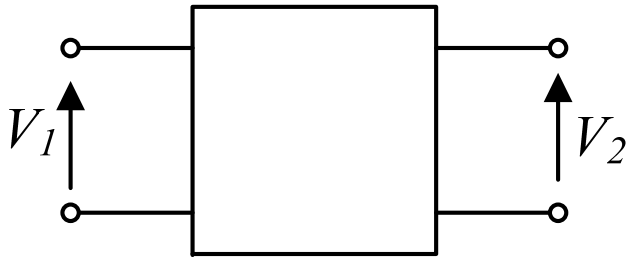
# Transfer function in frequency domain

- The frequency domain transfer function (or the frequency response) is obtained from the transfer function under the condition that  $s = j\omega$ .
- The frequency transfer function means the **steady state characteristic** of the circuit which is stimulated with  $e^{j\omega t}$ .

$$H(s) = \frac{a}{b+s} \xrightarrow{s=j\omega} H(j\omega) = \frac{a}{b+j\omega} = \frac{\frac{a}{b}}{1+j\frac{\omega}{b}}$$



# Two-port network parameters and transfer function

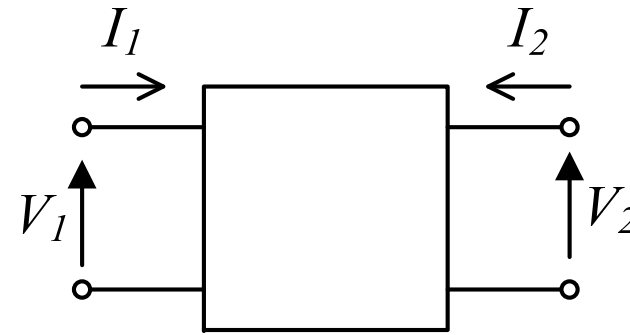


Transfer function

$$V_2(s) = H(s) \cdot V_1(s)$$

The transfer function varies in according with the load impedance.

Two-port network parameters do not vary in according with the load impedance.



Two-port network parameters

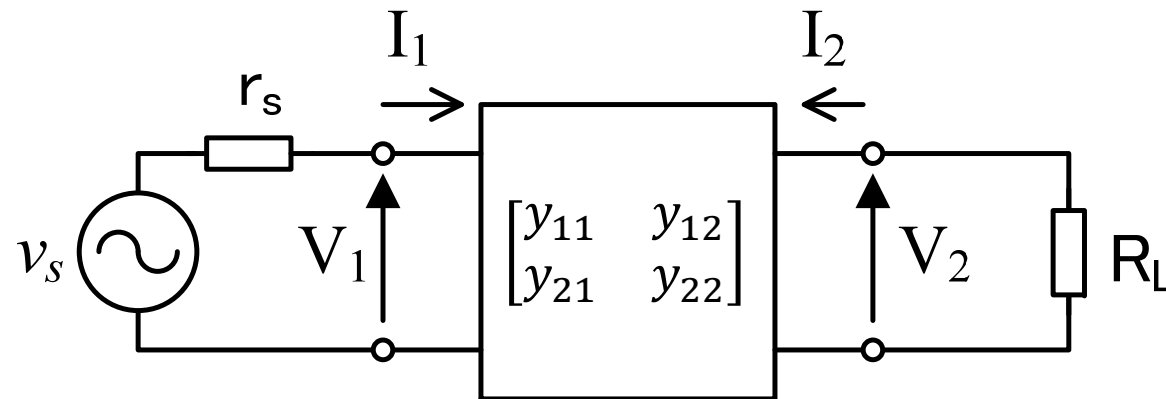
$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ I_1(j\omega) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(j\omega) \\ -I_2(j\omega) \end{bmatrix}$$



# Load-dependence of transfer function

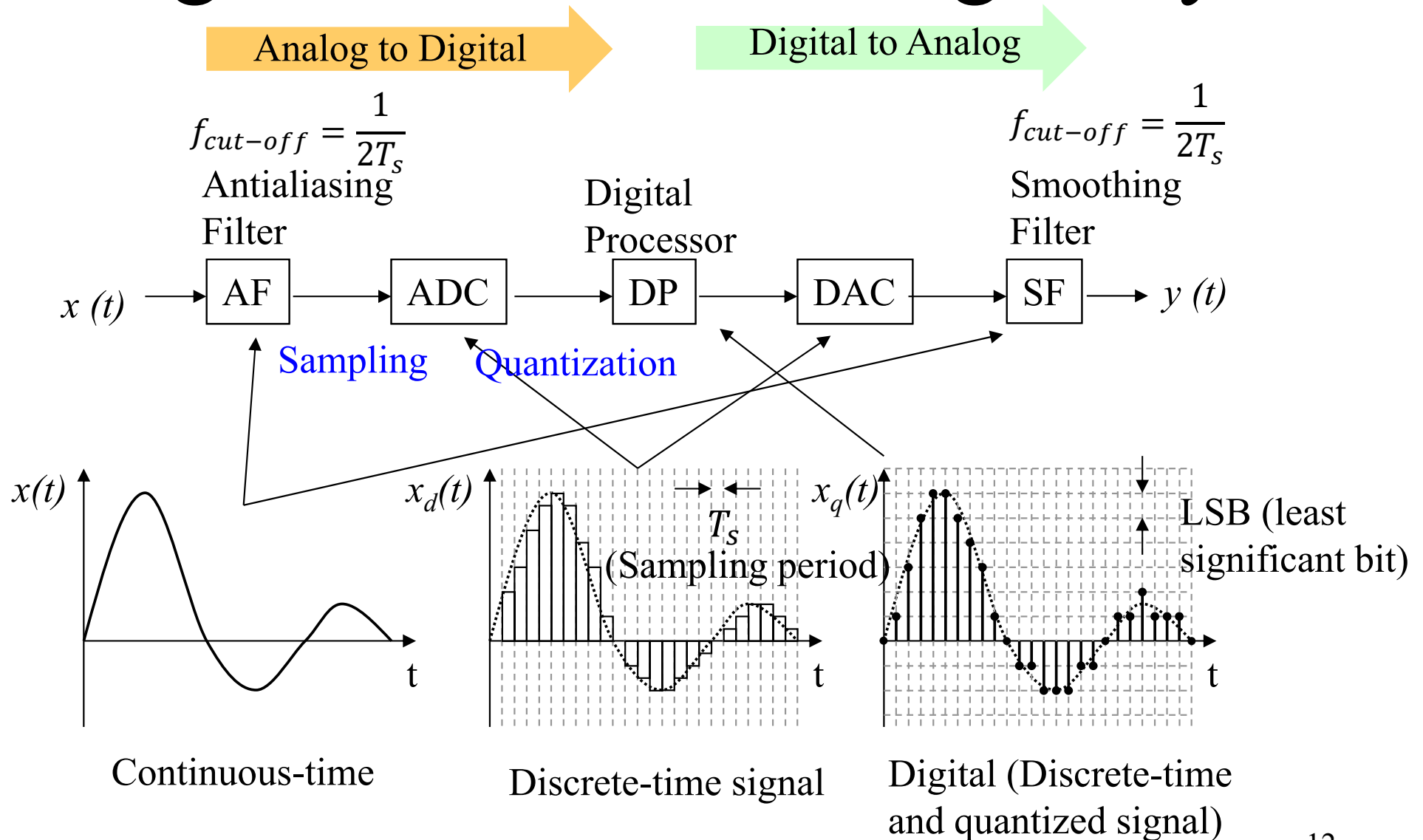


$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} \Rightarrow \begin{cases} I_2 = y_{21}V_1 + y_{22}V_2 \\ -V_2 = R_L I_2 \end{cases}$$

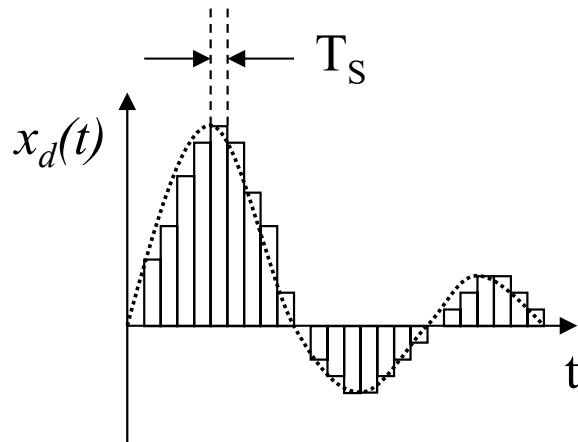
$$H(s = j\omega) = \frac{V_2}{V_1} = -\frac{y_{21}}{y_{22} + \frac{1}{R_L}}$$

## 2.2 Waveforms in mixed-signal circuits

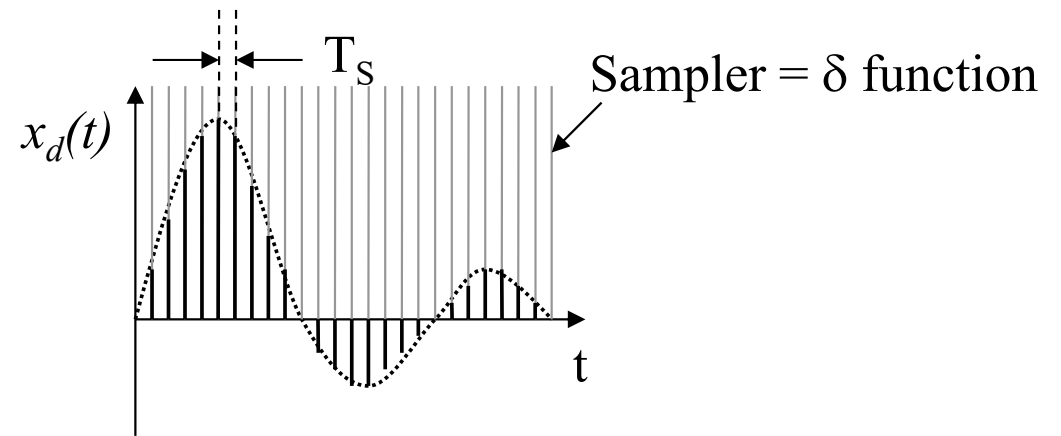
# Configuration of mixed-signal system



# Expressions of discrete-time signal



Step sampling  
(Zero-order hold)



Impulse sampling  
(Pulse Amplitude Modulation or PAM)

$$x_{du}(t) = \sum_n x(nT_s) \cdot \underbrace{\{u(t - nT_s) - u(t - (n+1)T_s)\}}_{\text{Step Sampler}}$$

Step Sampler

$$x_d(t) = \sum_n x(t) \cdot \underbrace{\delta(t - nT_s)}_{\text{Impulse Sampler}}$$

Impulse Sampler

NOTE: The discrete analog signal is practically obtained by S/H, but the signal can be handled similar to the impulse sequences (See slide 11).

# Laplace transform and Z transform

Continuous-time signal  $x(t)$   $\xrightarrow{\mathcal{L}}$   $X(s) = \int_0^{\infty} x(t)e^{-st} dt$

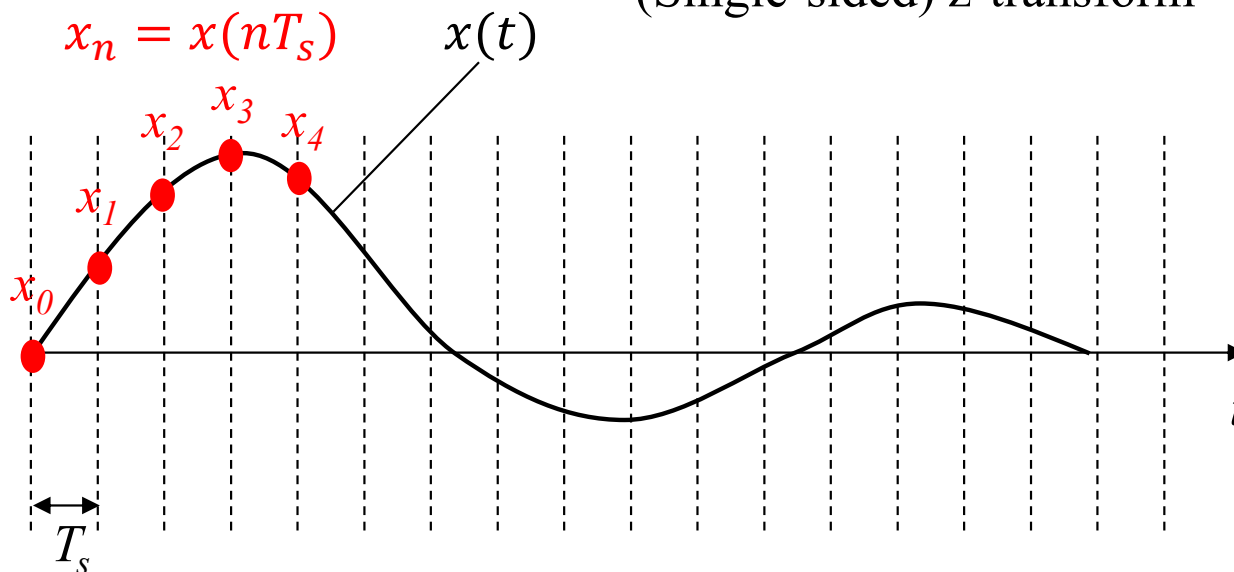
Discrete-time signal  $x(t)$   $\xrightarrow{\mathcal{Z}}$   $X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

(Single-sided) z-transform

$$s = \sigma + j\omega$$



$$z^{-1} = e^{-sT_s}$$





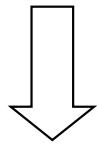
# Characteristic of $\delta$ function

Step function  $f(t)$

$$f(t) = \frac{1}{T_P} \left\{ u\left(t + \frac{T_P}{2}\right) - u\left(t - \frac{T_P}{2}\right) \right\}$$

$$\int_{-\infty}^{\infty} f(t) dt = 1 \quad (1)$$

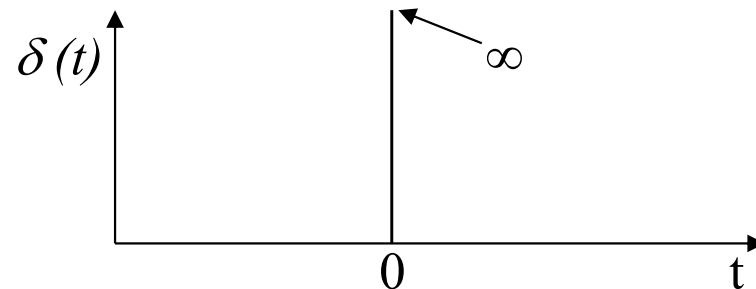
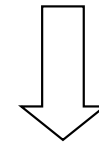
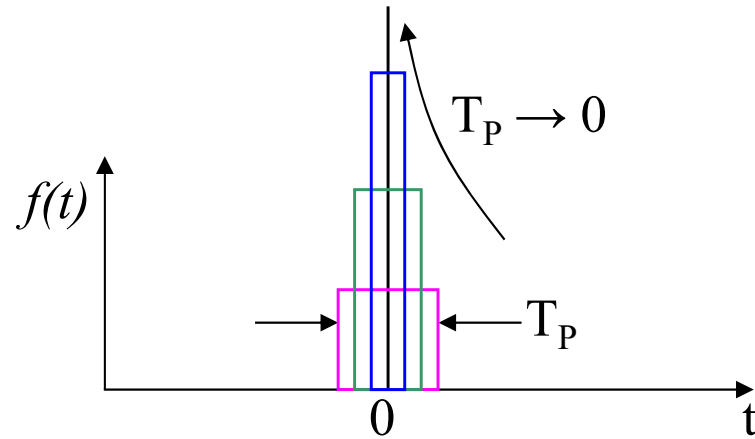
$$\delta(t) \equiv \lim_{T_P \rightarrow 0} f(t) \quad (2)$$



Delta function  $\delta$

$$\left\{ \begin{array}{l} \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3) \\ \delta(t) = 0 \quad (t \neq 0) \quad (4) \end{array} \right.$$

Important



# Z transform of discrete-time signal

PAM signal

$$x_d(t) = \sum_n x(nT_s) \delta(t - nT_s)$$

↓ Laplace transform

$$\begin{aligned} X_d(s) &= \int_0^\infty x_d(t) e^{-st} dt \\ &= \int_0^\infty \sum_n x(nT_s) \delta(t - nT_s) e^{-st} dt \\ &= \sum_n \int_0^\infty x(nT_s) \delta(t - nT_s) e^{-st} dt \\ &= \sum_n x(nT_s) e^{-s(nT_s)} \\ &= \sum_n x(nT_s) z^{-n} \end{aligned}$$

$\left. \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \right\} z^{-1} = e^{-sT_s}$

Zero-order hold signal

$$x_u(t) = \sum_n x(nT_s) \{u(t - nT_s) - u(t - (n+1)T_s)\}$$

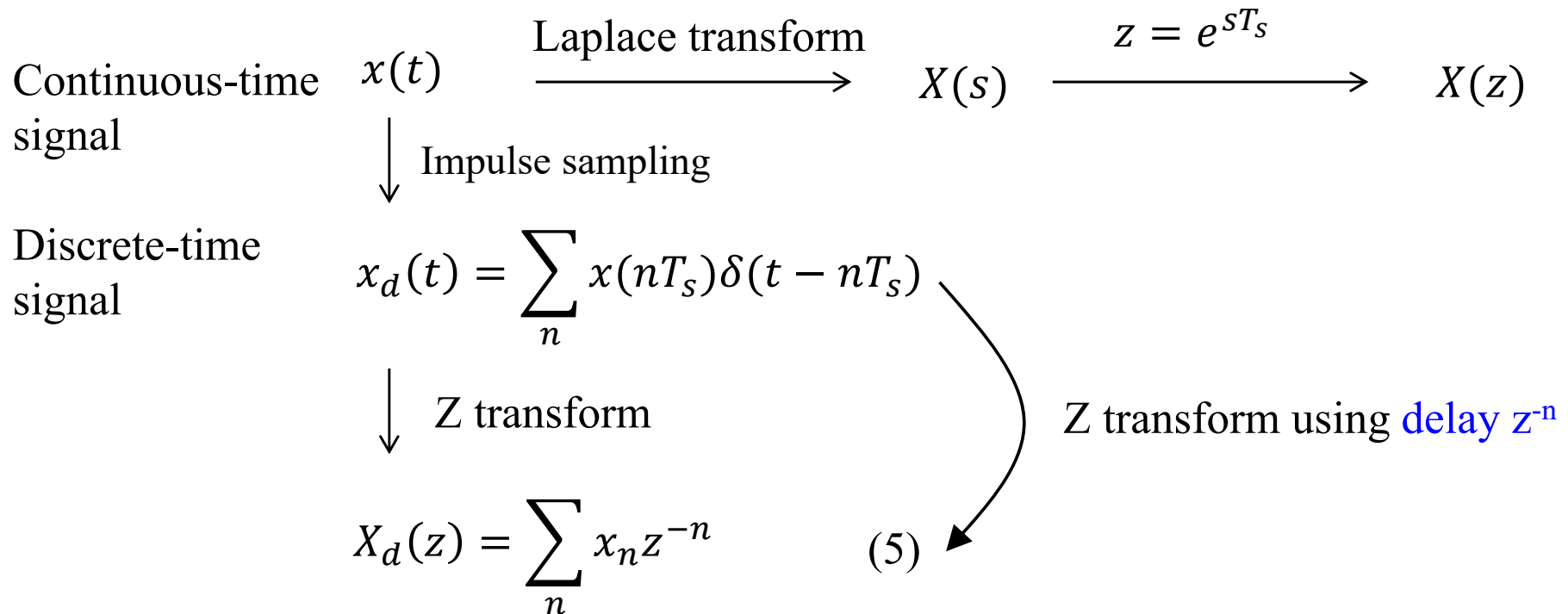
↓ Laplace transform

$$\begin{aligned} X_u(s) &= \int_0^\infty x_u(t) e^{-st} dt \\ &= \int_0^\infty \sum_n x(nT_s) \{u(t - nT_s) - u(t - (n+1)T_s)\} e^{-st} dt \\ &= \sum_n x(nT_s) \int_0^\infty \{u(t - nT_s) - u(t - (n+1)T_s)\} e^{-st} dt \\ &= \sum_n x(nT_s) \frac{e^{-snT_s}(1 - e^{-sT_s})}{s} \\ &= \frac{1 - e^{-sT_s}}{s} \sum_n x(nT_s) z^{-n} \end{aligned}$$

$\left. \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \right\} z^{-1} = e^{-sT_s}$

See Appendix 2(3). ← Transfer function of step sampler      PAM signal

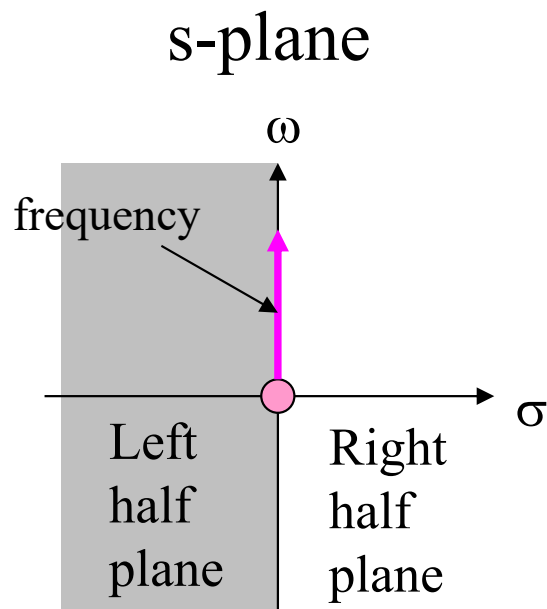
# Z transform of PAM signal



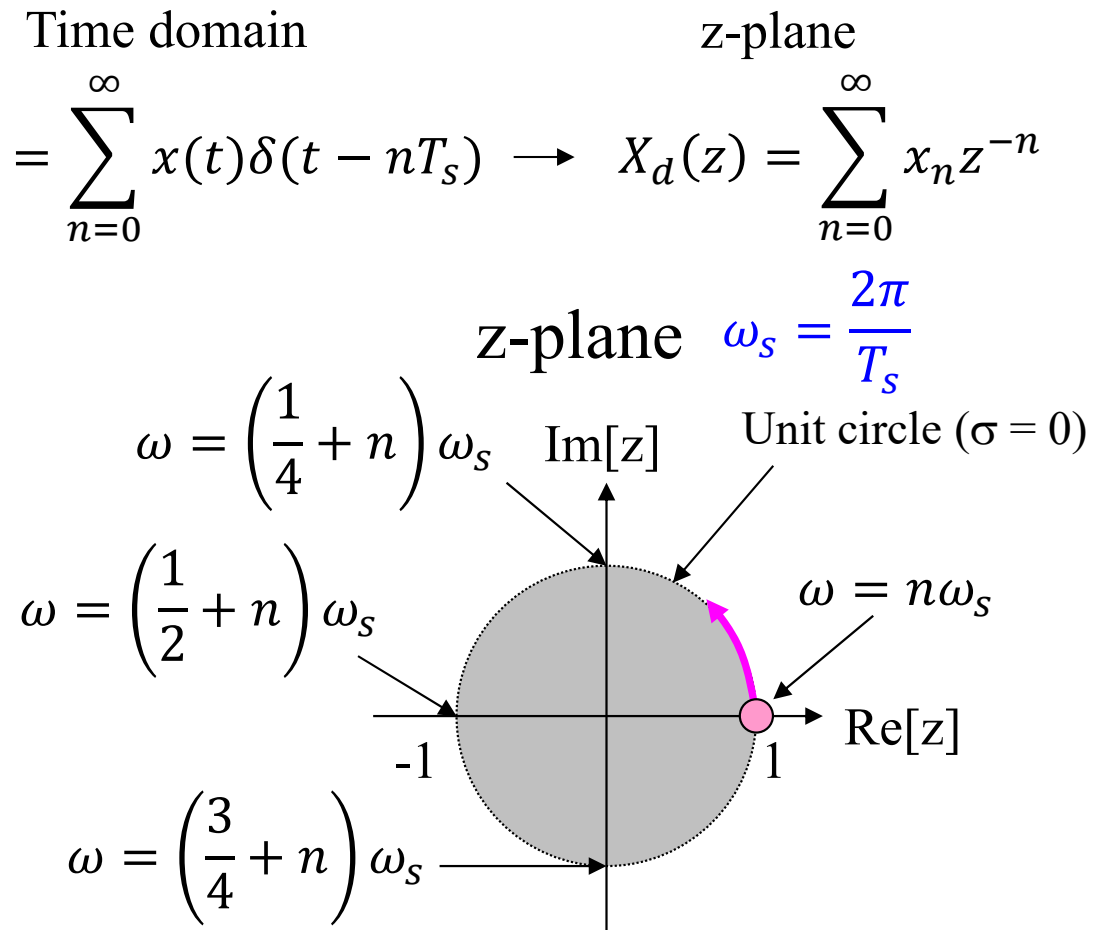
A discrete-time signal can be easily transformed by using Eq.(5). You do not need to calculate by Laplace transform.

# s-plane and z-plane

$$X_d(s) = \sum_n x(nT_s) e^{-s(nT_s)} \leftarrow x_d(t) = \sum_{n=0}^{\infty} x(t) \delta(t - nT_s) \rightarrow X_d(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$



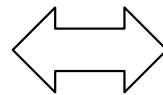
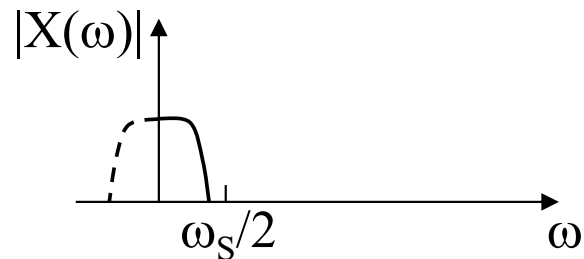
$$s = \sigma + j\omega$$



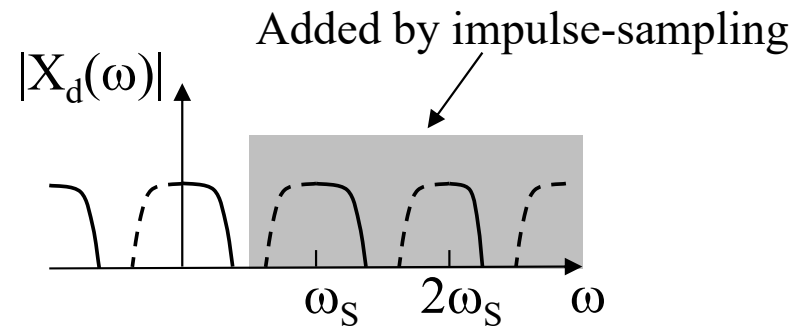
$$\begin{aligned} z &= e^{sT_s} = e^{\sigma T_s} e^{j\omega T_s} \\ &= e^{\sigma T_s} (\cos \omega T_s + j \sin \omega T_s) \\ &= e^{\sigma T_s} (\cos \omega_s + j \sin \omega_s) \end{aligned}$$

# Spectrum of PAM signal

Spectrum of continuous-time signal

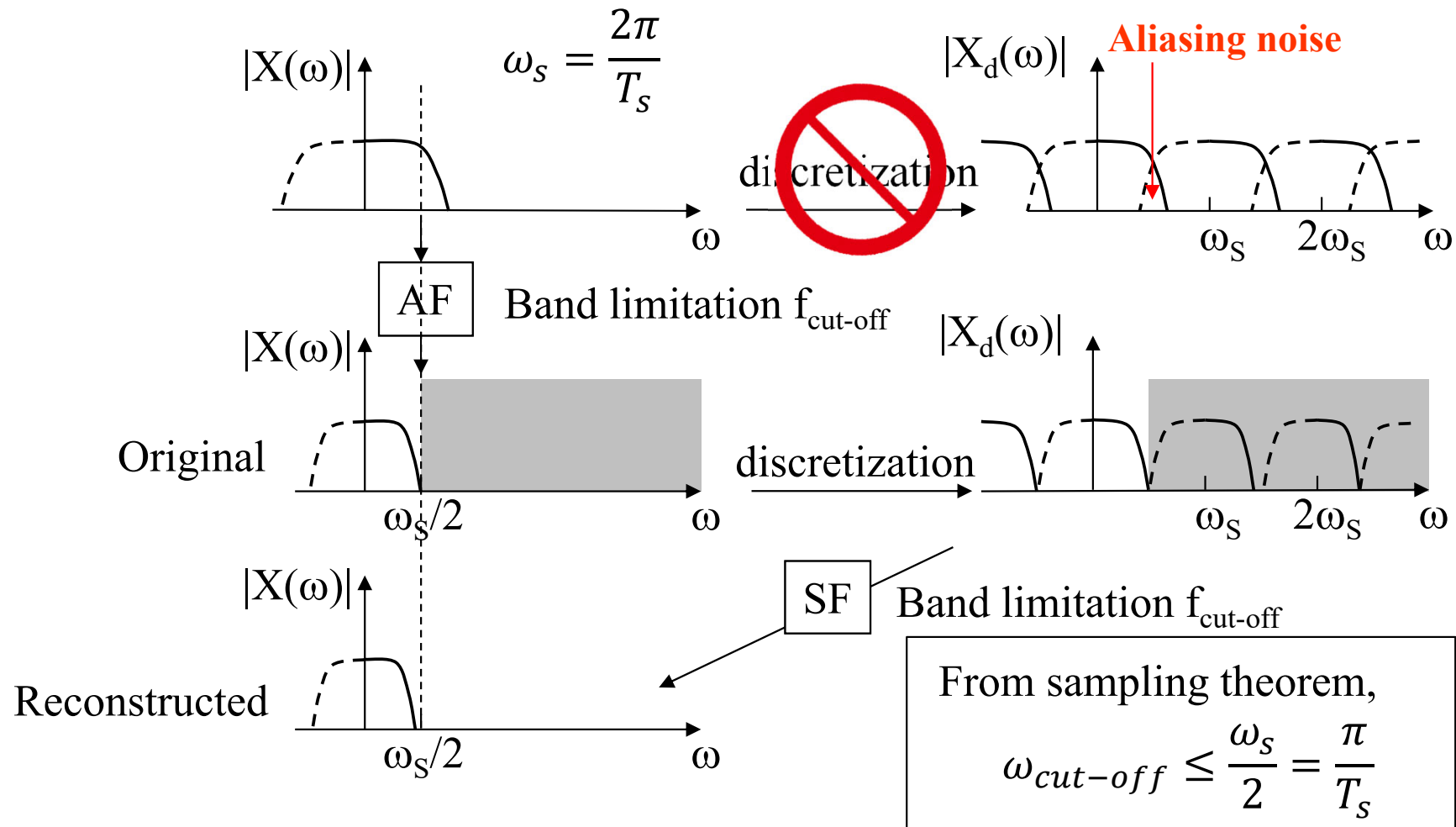


Spectrum of discrete-time signal



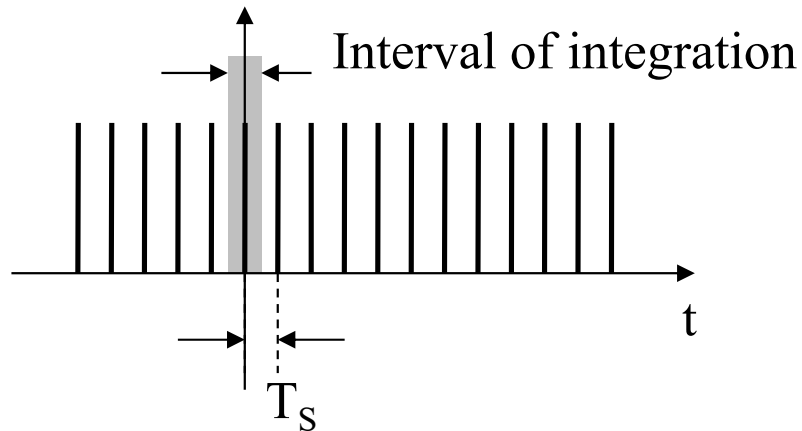
See Appendix 2(1) and 2(2).

# Antialiasing and smoothing of signals



NOTE: AF and SF can be implemented in continuous-time circuits.

# Appendix 2(1) Fourier series of impulse sampler



$$\delta_T(t) = \sum_n \delta(t - nT_S)$$

$$\delta_T(t) = \sum_n c_n e^{jn\frac{2\pi}{T_S}t}$$

$$c_n = \frac{1}{T_S} \int_{-\frac{T_S}{2}}^{\frac{T_S}{2}} \delta_T(t) \cdot e^{jn\frac{2\pi}{T_S}t} dt$$

$$= \frac{1}{T_S} e^{jn\frac{2\pi}{T_S}0} = \frac{1}{T_S}$$

$$\therefore \delta_T(t) = \frac{1}{T_S} \sum_n e^{jn\frac{2\pi}{T_S}t}$$

# Appendix 2(2) Spectrum of PAM

$$x_d(t) = \sum_n x(t) \cdot \delta(t - nT_S) = x(t) \cdot \delta_T(t) = \frac{1}{T_S} \sum_n x(t) \cdot e^{jn\frac{2\pi}{T_S}t}$$

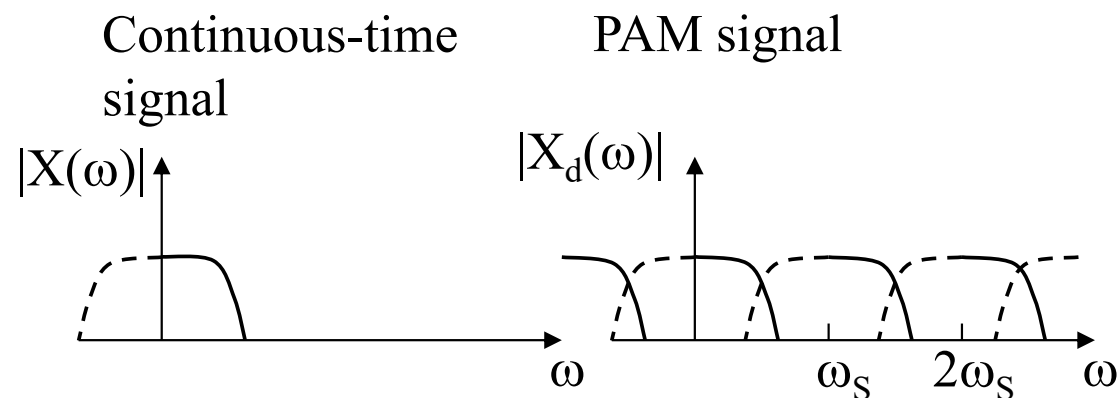
$\downarrow$  Translation Theorem  $\mathcal{L}[e^{-at}f(t)] = F(s + a)$

$$X_d(s) = \frac{1}{T_S} \sum_n X(s - jn\frac{2\pi}{T_S}) = \frac{1}{T_S} \sum_n X(s - jn\omega_S)$$

$$\omega_S = \frac{2\pi}{T_S}$$

Let's say,  $s = j\omega$

$$X_d(\omega) = \frac{1}{T_S} \sum_n X\{j(\omega - n\omega_S)\}$$



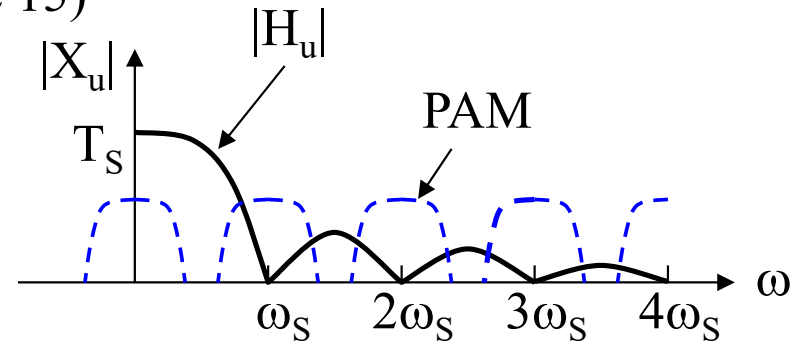


# Appendix 2(3) Spectrum of zero-order hold

Laplace transform of zero-order hold signal (Slide 15)

$$X_u(s) = \frac{1 - s^{-sT_s}}{s} \underbrace{\sum_n x(nT_s)z^{-n}}_{\text{PAM signal}}$$

Transfer function of step sampler  $H_u(s)$



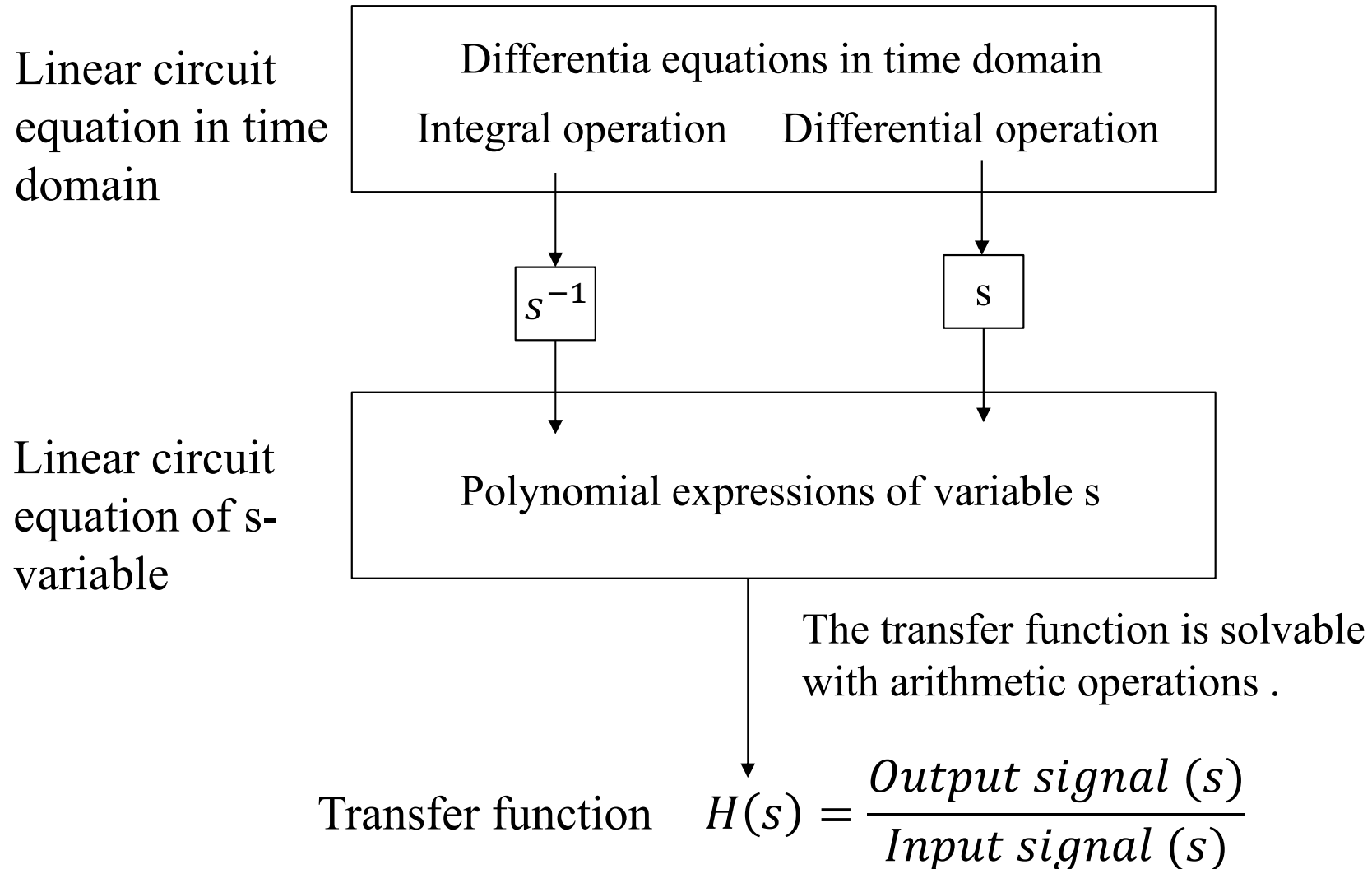
$$H_u(s) = \frac{1 - e^{-sT_s}}{s} = e^{-\frac{sT_s}{2}} \frac{e^{\frac{sT_s}{2}} - e^{-\frac{sT_s}{2}}}{s} \xrightarrow{s=j\omega} e^{-j\frac{\omega T_s}{2}} \frac{e^{j\frac{\omega T_s}{2}} - e^{-j\frac{\omega T_s}{2}}}{j\omega} = e^{-j\frac{\omega T_s}{2}} T_s \frac{\sin \frac{\omega T_s}{2}}{\frac{\omega T_s}{2}}$$

$$= \underbrace{e^{-j\frac{\omega T_s}{2}}}_{\text{Phase}} \underbrace{T_s \text{Sinc}\left(\frac{\omega T_s}{2}\right)}_{\text{Amplitude}}$$

NOTE: The spectrum of zero-order hold signal is deviated from the spectrum of PAM signal by the step sampler  $H_u(s)$ . Therefore, the smoothing filter after DAC must have the  $\text{Sinc}^{-1}$  characteristic.

## 2.3 Transfer function of continuous-time analog circuits

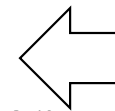
# Integration and Differentiation



# Definition of transfer function

- Transfer function

- $s = \sigma + j\omega$  : Transfer function
- $s = j\omega$  : Frequency domain transfer function



$$H(s) = \frac{\text{Output signal } (s)}{\text{Input signal } (s)}$$

- Pole and Zero

- $1/H(s_p) = 0$ , for the complex number  $s_p$  at a location of pole in s-plane
- $H(s_z) = 0$ , for the complex number  $s_z$  at a location of zero in s-plane

- Corner frequency of pole and zero

- A corner frequency in Bode diagram is observed as a consequence of pole and zero.
- Pole frequency: The corner of amplitude response is convex downward.
- Zero frequency: The corner of amplitude response is convex upward.

# Decibel (dB)

The vertical axis of Bode diagrams is plotted in the decibel scale (dB). The decibel indicates the absolute value ratio of the signal amplitude.

Decibel of voltage and current signal  $dB = 20 \log_{10} \left| \frac{V_2}{V_1} \right| = 20 \log_{10} |H(\omega)|$

Decibel of signal power  $dB = 10 \log_{10} \left| \frac{P_2}{P_1} \right|$

Note:  $dBm$  is not ratio, but the absolute value of the signal power in mW.

$$dBm = 10 \log_{10} P(mW)$$

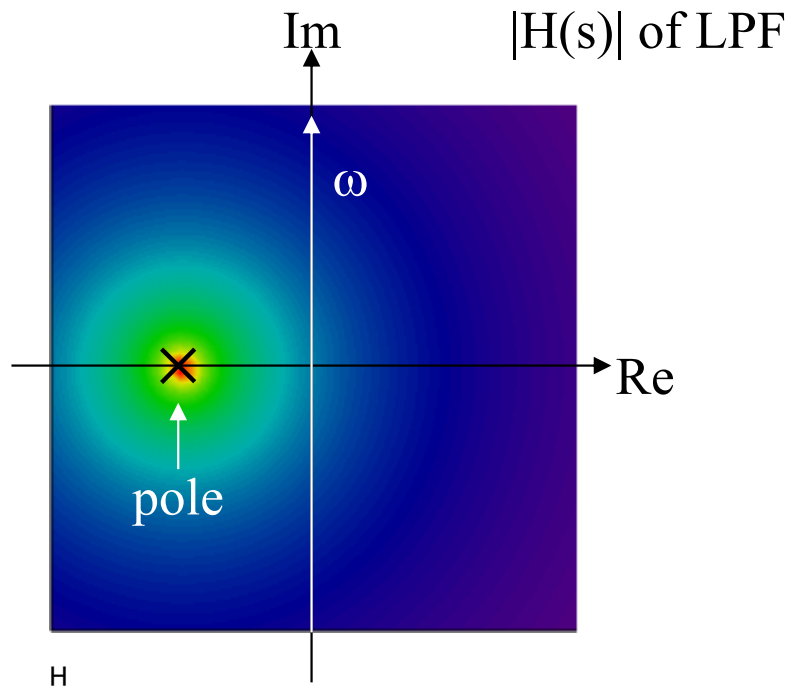
# 1-pole transfer function

$$H(s) = \frac{a \cdot s + b}{s + c} = \frac{b}{c} \frac{\left(1 + \frac{as}{b}\right)}{\left(1 + \frac{s}{c}\right)}$$

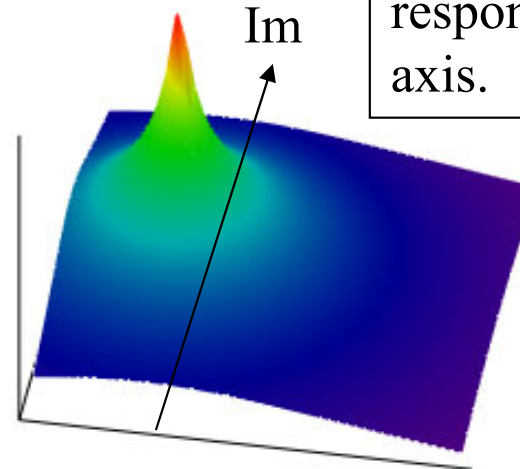
(a, b, c = real number)

Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF



$$H(s) = \frac{b}{s + c}$$

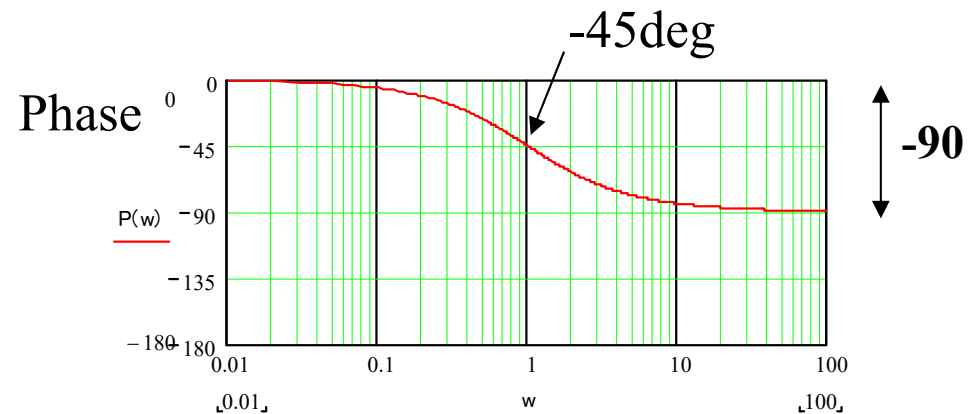
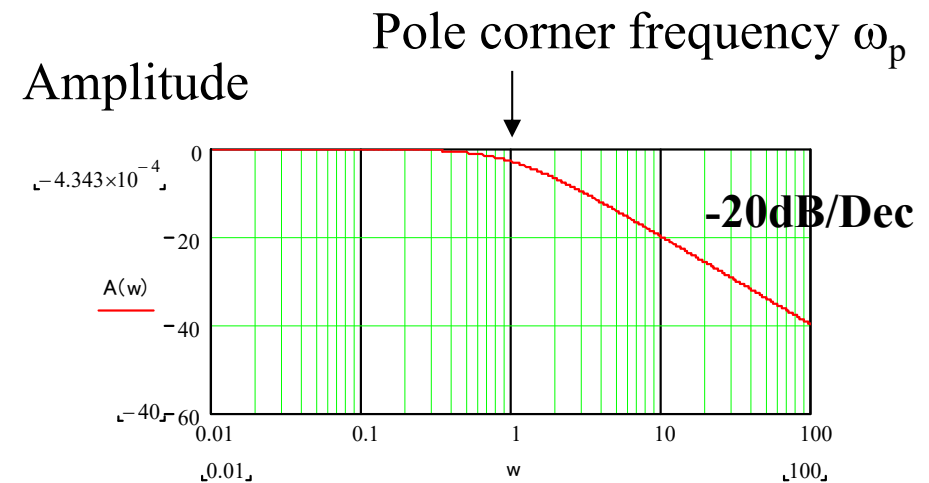
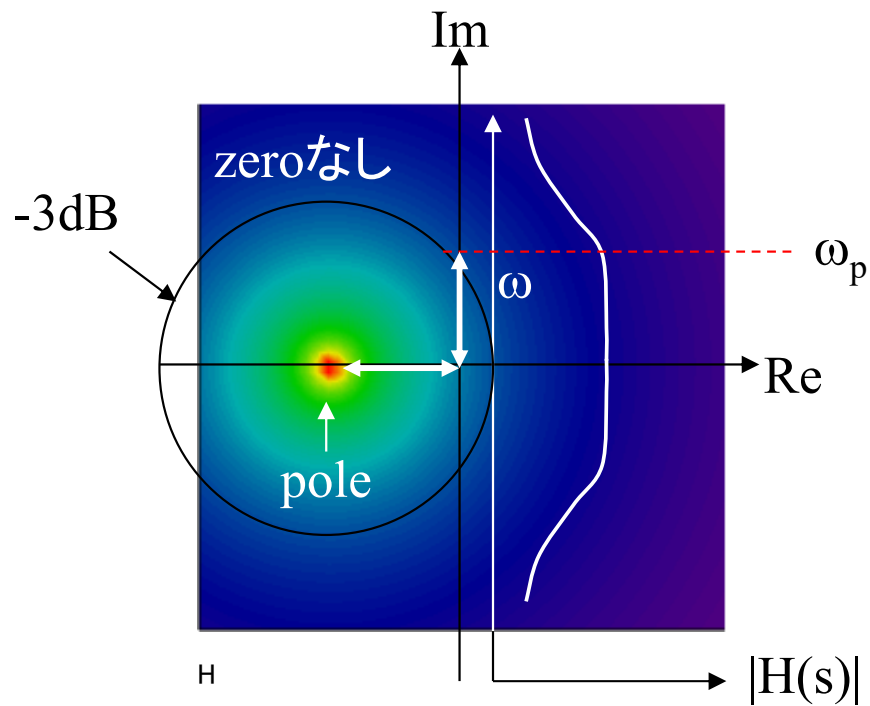


The Bode diagram is represents the amplitude response in the imaginary axis.

# Bode diagram of 1st order LPF

Transfer function

$$H(s) = \frac{b}{s + c}$$

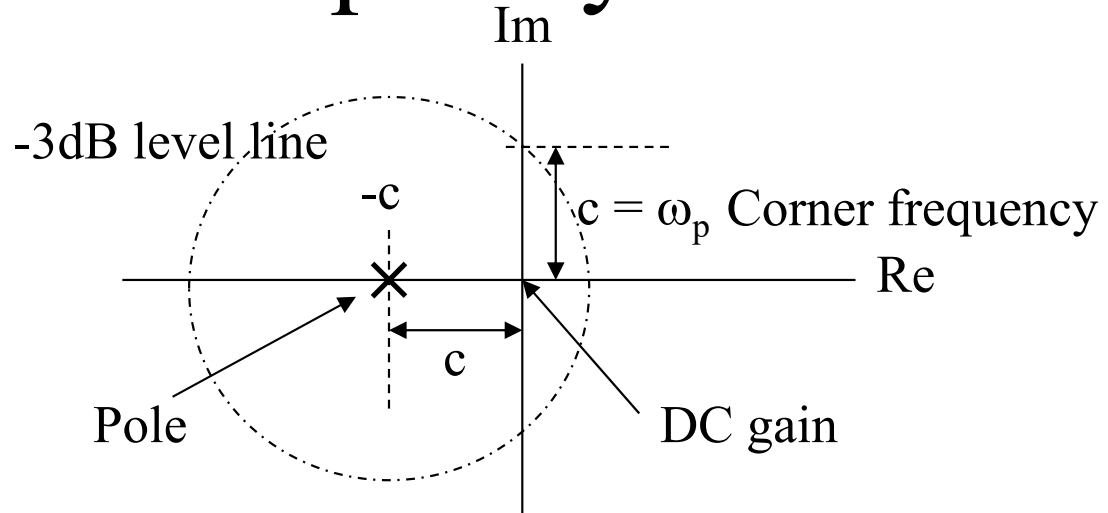


# Positional relation between pole and corner frequency

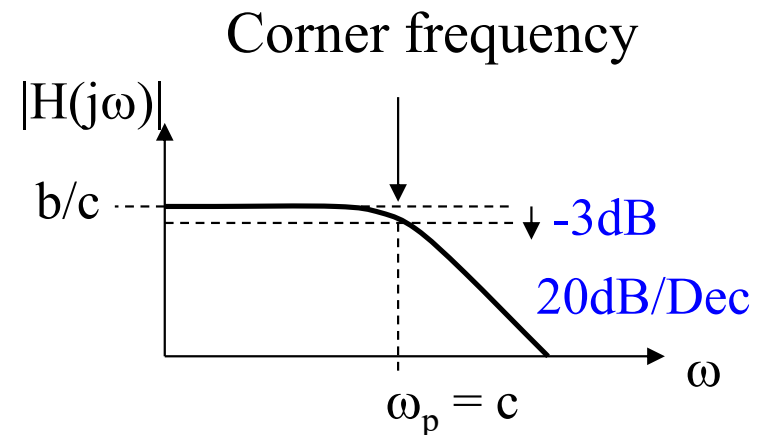
$$H(s) = \frac{b}{s+c}$$

$$H(j\omega) = \frac{\frac{b}{c}}{1+j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\frac{b}{c}}{\sqrt{1+\frac{\omega^2}{c^2}}}$$



$$\left\{ \begin{array}{l} \omega \ll c \rightarrow |H(j\omega)| = \frac{b}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}} \frac{b}{c} \\ \omega \gg c \rightarrow |H(j\omega)| = \frac{b}{\omega} \end{array} \right. \quad \frac{1}{\sqrt{2}} = -3\text{dB}$$





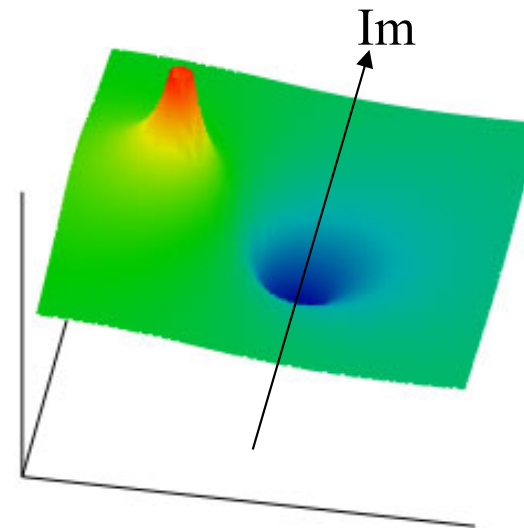
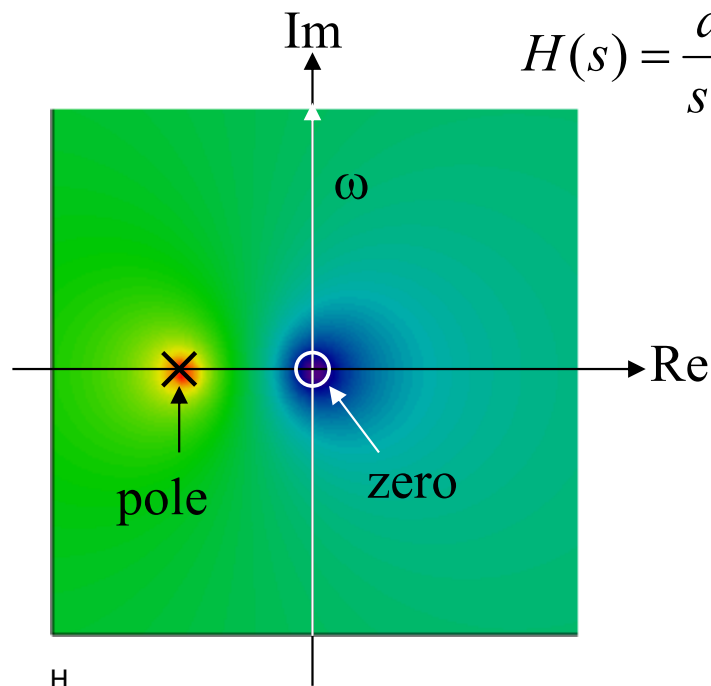
# 1-pole, 1-zero transfer function

$$H(s) = \frac{a \cdot s + b}{s + c} \quad (a, b, c = \text{real number})$$

Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF

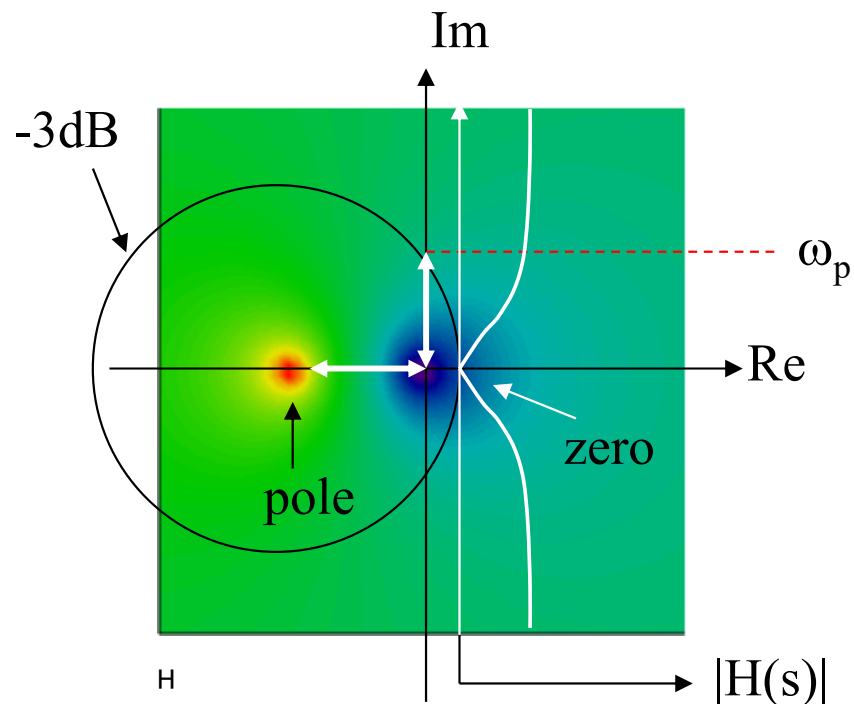
$|H(s)|$  of HPF



# Bode diagram of 1st order HPF

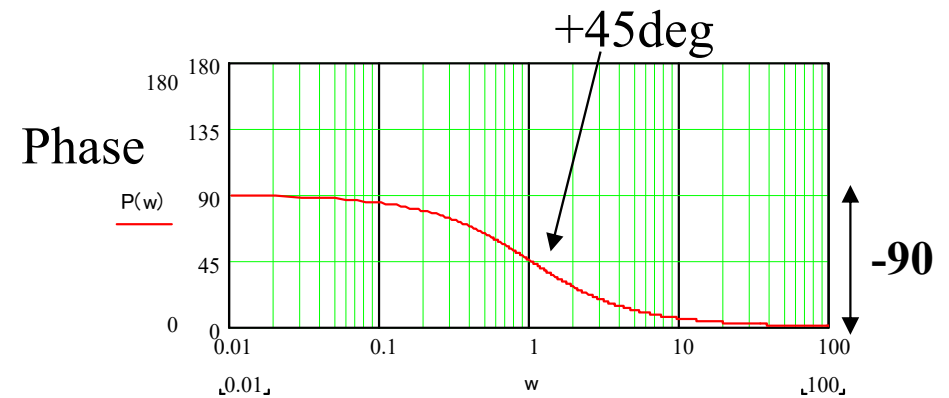
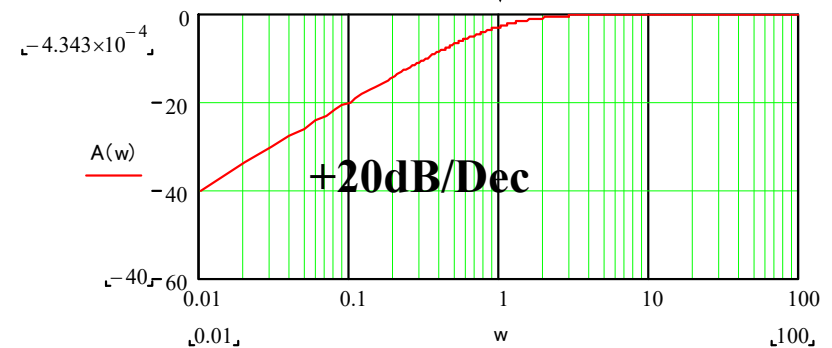
Transfer function

$$H(s) = \frac{a \cdot s}{s + c}$$



Amplitude

Pole corner frequency  $\omega_p$

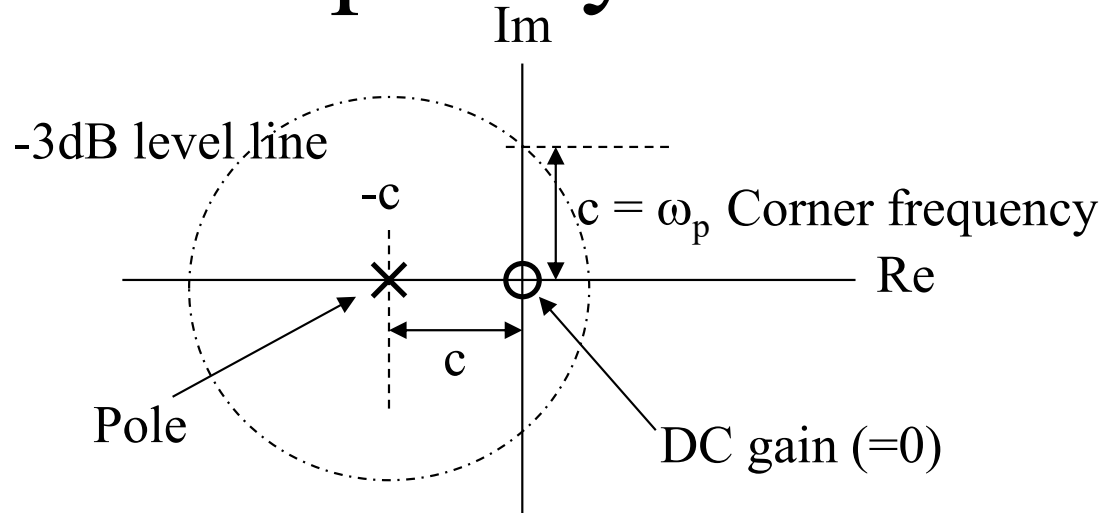


# Positional relation between pole and corner frequency

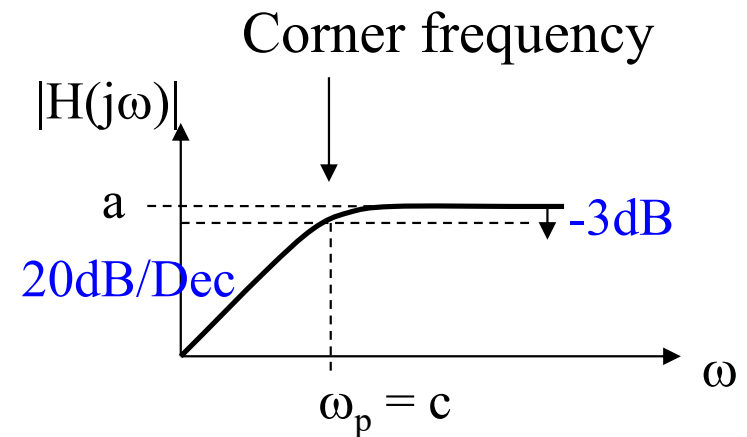
$$H(s) = \frac{as}{s+c}$$

$$H(j\omega) = \frac{j\omega \frac{a}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\omega \frac{a}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\left\{ \begin{array}{l} \omega \ll c \rightarrow |H(j\omega)| = \omega \frac{a}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{a}{\sqrt{2}} \\ \omega \gg c \rightarrow |H(j\omega)| = a \end{array} \right. \quad \frac{1}{\sqrt{2}} = -3\text{dB}$$



# 2-pole transfer function

$$H(s) = \frac{a \cdot s^2 + b \cdot s + c}{s^2 + d \cdot s + e}$$

(a, b, c = real number)

|H(s)| of LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

Solution of the pole

$$D(s) = s^2 + d \cdot s + e = 0$$

$$s = -\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}}$$

Complex number of 2 poles

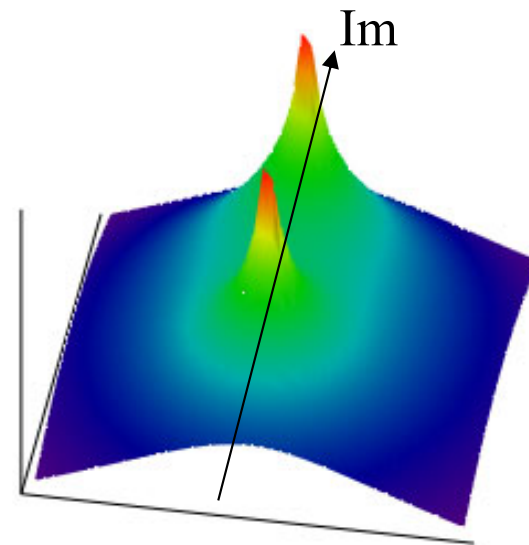
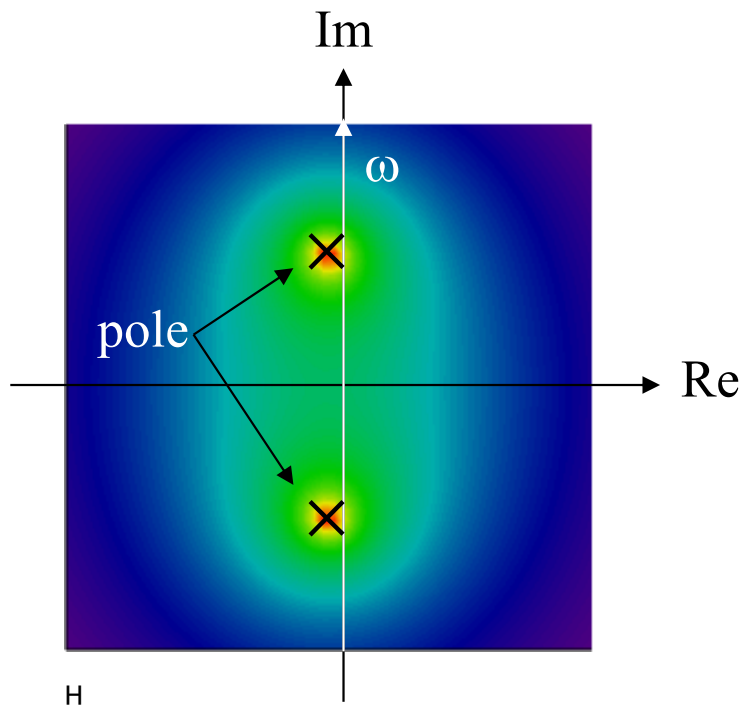
Type of frequency response

$a = b = 0, \quad c \neq 0$	LPF
$a = c = 0, \quad b \neq 0$	BPF
$b = c = 0, \quad a \neq 0$	HPF
$b = 0, \quad a \neq 0, \quad c \neq 0$	BEF

Note: If the denominator is factorable, the transfer function has 2 real poles.

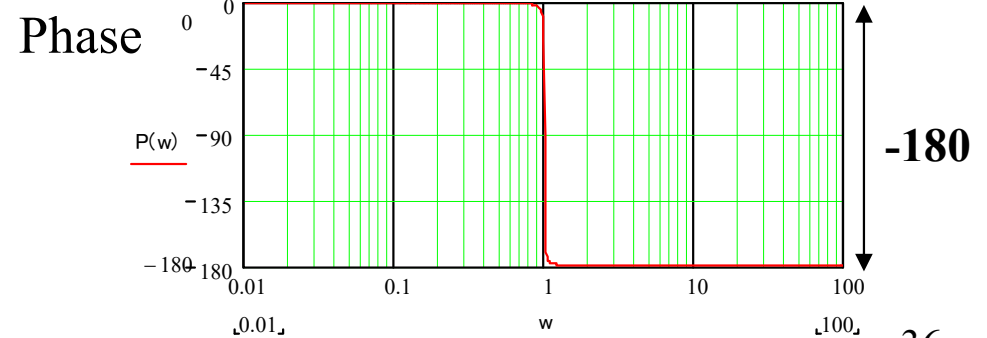
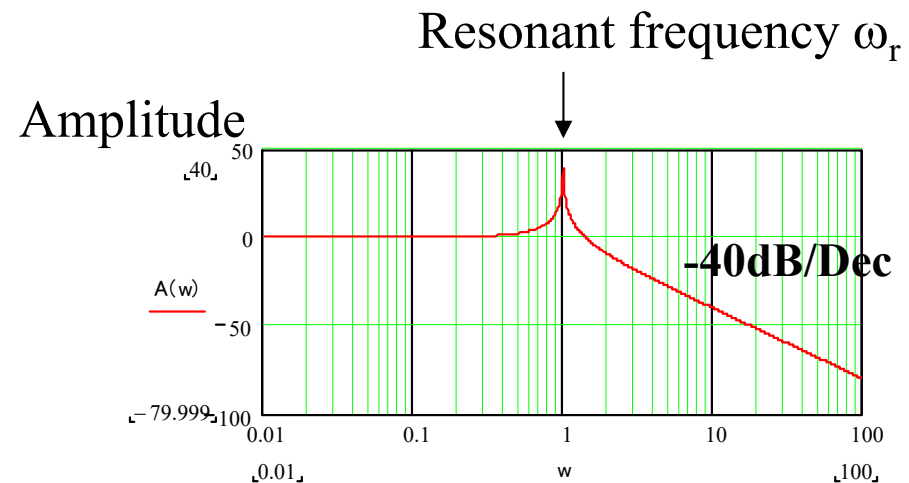
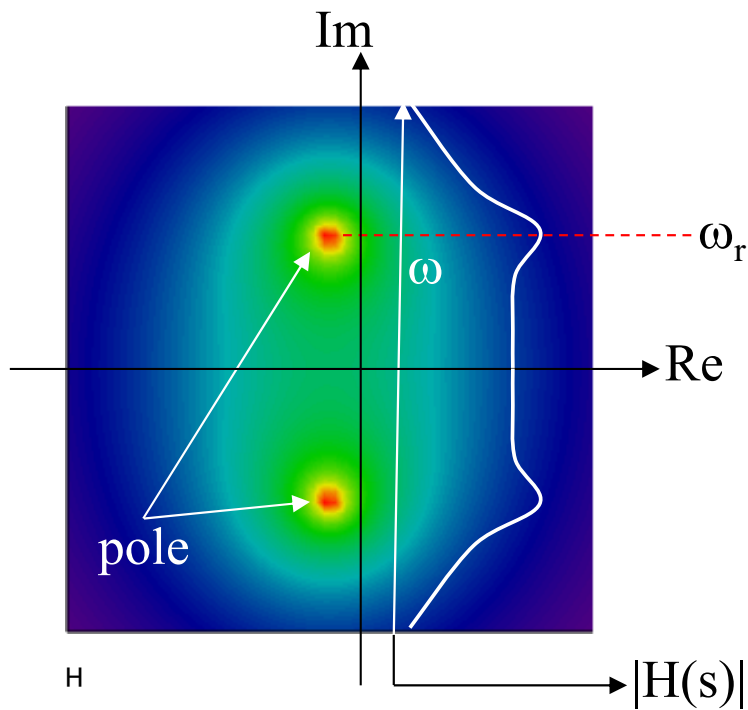
# Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e} \quad \text{の場合}$$



# Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$



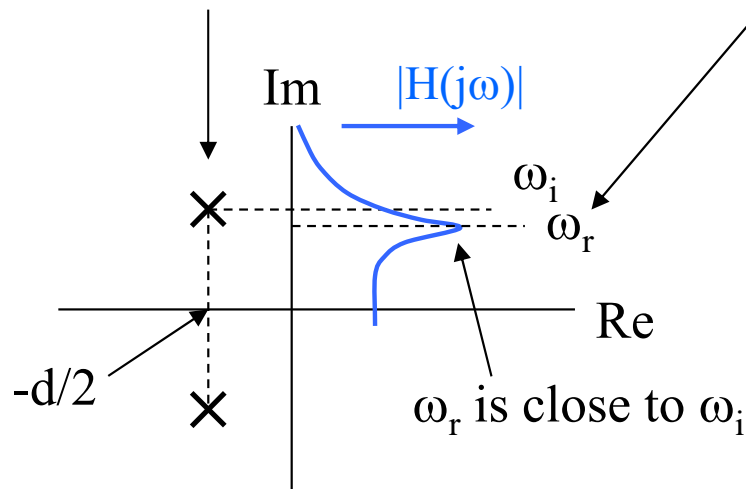
# Positional relation between pole and corner frequency

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

$s^2 + d \cdot s + e = 0$  のとき

$$s = -\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}} \equiv -\frac{d}{2} \pm j\omega_i$$

Location of the pole in s-plane



$$|H(j\omega)| = \frac{c}{\sqrt{\{(j\omega)^2 + d(j\omega) + e\} \{(-j\omega)^2 + d(-j\omega) + e\}}}$$

$$= \frac{c}{\sqrt{\{\omega^2 - (e - \frac{d^2}{4})\}^2 + d^2(e - \frac{d^2}{4})}}$$

$$\omega_r \equiv \sqrt{e - \frac{d^2}{4}}$$

$$|H(j\omega)| = \frac{c}{\sqrt{\{\omega^2 - \omega_r^2\}^2 + d^2(e - \frac{d^2}{4})}}$$

If  $\omega = \omega_r$ , the amplitude reach a maximum.

$$|H(j\omega_r)| = \frac{c}{d\sqrt{e - \frac{d^2}{4}}}$$

The smaller d causes the higher peak.

## 2.4 Transfer function of discrete-time analog/digital circuits



# Definition of transfer function on z-plane

$$H(s) = \frac{\textit{Output signal (s)}}{\textit{Input signal (s)}}$$

$$z^{-1} = e^{-sT_s}$$

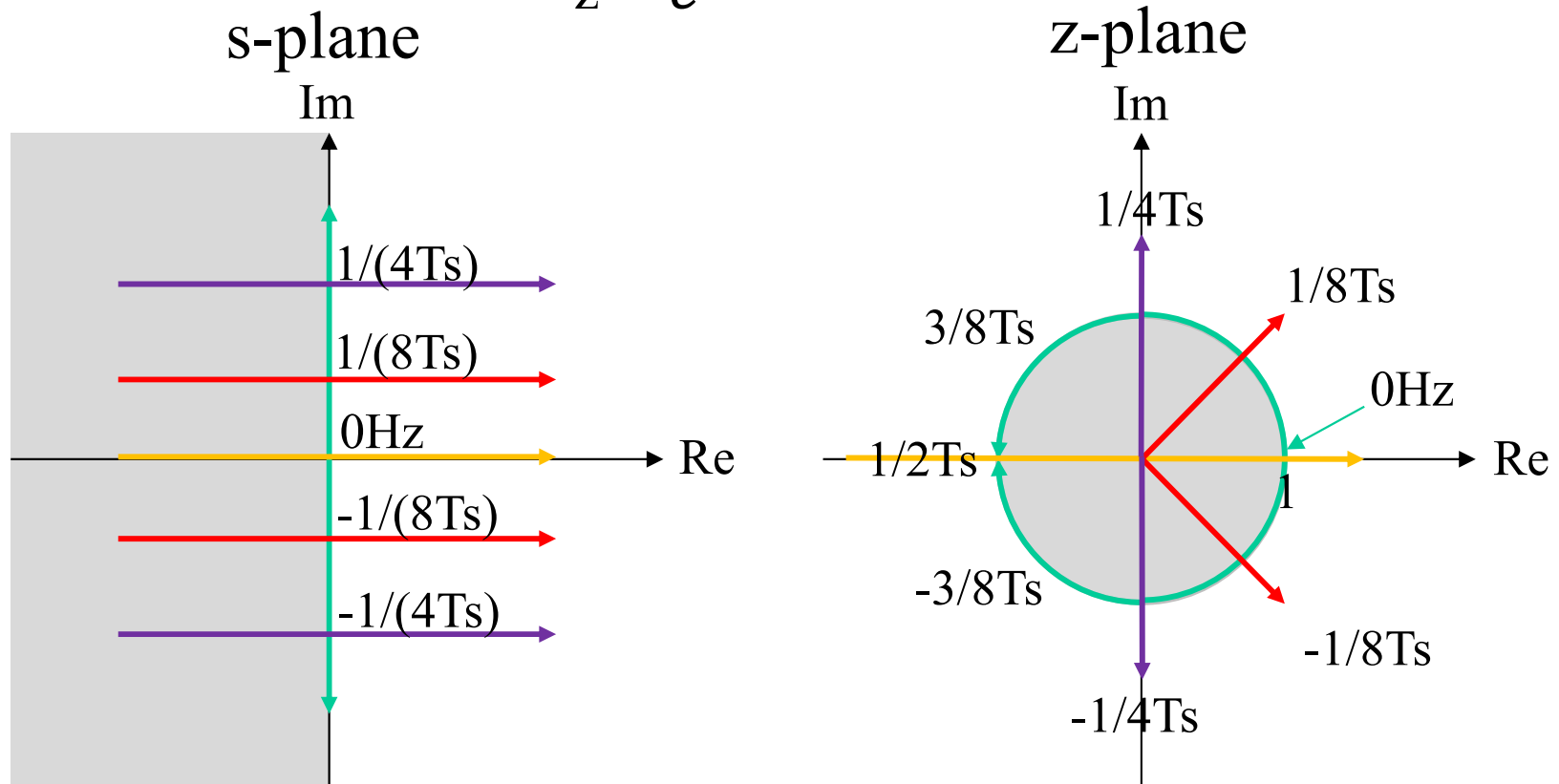


$$H(z) = \frac{\textit{Output signal (z)}}{\textit{Input signal (z)}}$$

The non-linear functions of z-variable are deduced from the rational functions including an integration and a differentiation of s-variable. The complexity of the circuit implementation is avoided by using a translation theorem and some approximations.

# Correspondence relation of s-domain and z-domain

$$z = e^{(\sigma + j\omega)T_s}$$



# Time shift in Laplace transform

$$k(t) = g(t - T)u(t - T) \quad , \text{ where } u(t) \text{ is a unit step function}$$

$$\downarrow \mathcal{L}$$

$$K(s) = \int_0^{\infty} g(t - T)u(t - T)e^{-st} dt$$

$$\tau = t - T$$

$$K(s) = \int_{-T}^{\infty} g(\tau)u(\tau)e^{-s(\tau+T)} d\tau = e^{-sT} \int_0^{\infty} g(\tau)u(\tau)e^{-s\tau} d\tau$$

$$= e^{-sT} \int_0^{\infty} f(\tau)e^{-s\tau} d\tau = e^{-sT} F(s) \quad \leftarrow f(t) = g(t)u(t)$$

$$\downarrow \mathcal{Z} \quad \text{if } T = nT_s \quad (T_s: \text{Sampling period})$$

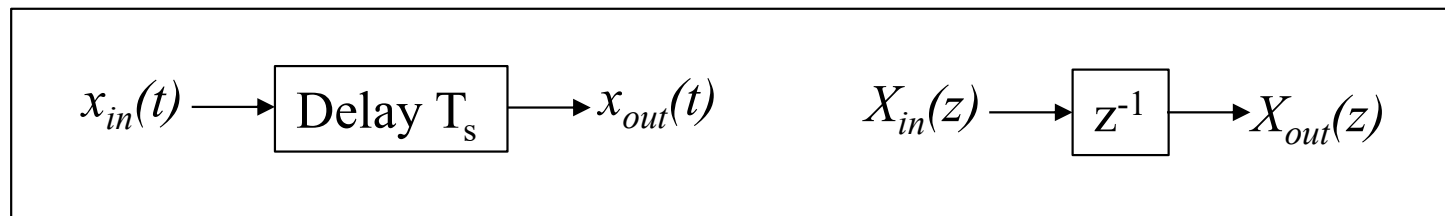
$$K(z) = e^{-snT_s} F(z) = z^{-n} F(z)$$

# Time shift in Z transform

$$x(nT_s) \xrightarrow{Z} X(z) \quad , \text{ where } t = nT_s \quad (6)$$

$$\text{(Right shift)} \quad x(nT_s - nT_s) \xrightarrow{Z} z^{-n} X(z) \quad (7)$$

$$\text{(Left shift)} \quad x(nT_s + mT_s) \xrightarrow{Z} z^m X(z) - z^m \sum_{k=0}^{m-1} x(kT_s) z^{-k} \quad (8)$$



The equation 7 shows that the multiplication of  $z^{-n}$  in z-plane is equivalent to the delay of  $nT_s$  in time domain. The  $z^{-1}$  operator is called "**Delay element**".

# Approximation of Z transform 1

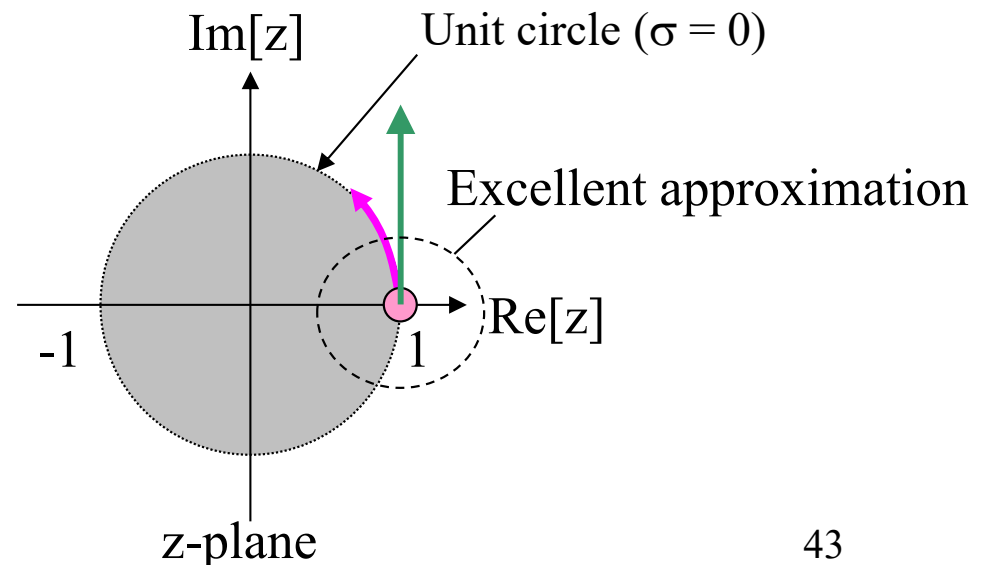
- Forward Euler Transformation (FET)

Power series expansion of  $z$  at  $s = 0$

$$z = e^{sT_s} = 1 + \frac{1}{1!}(sT_s)^1 + \frac{1}{2!}(sT_s)^2 + \frac{1}{3!}(sT_s)^3 + \Lambda$$

$$\approx 1 + sT_s$$

$$\therefore s \approx \frac{z-1}{T_s}$$



# Approximation of Z transform 2

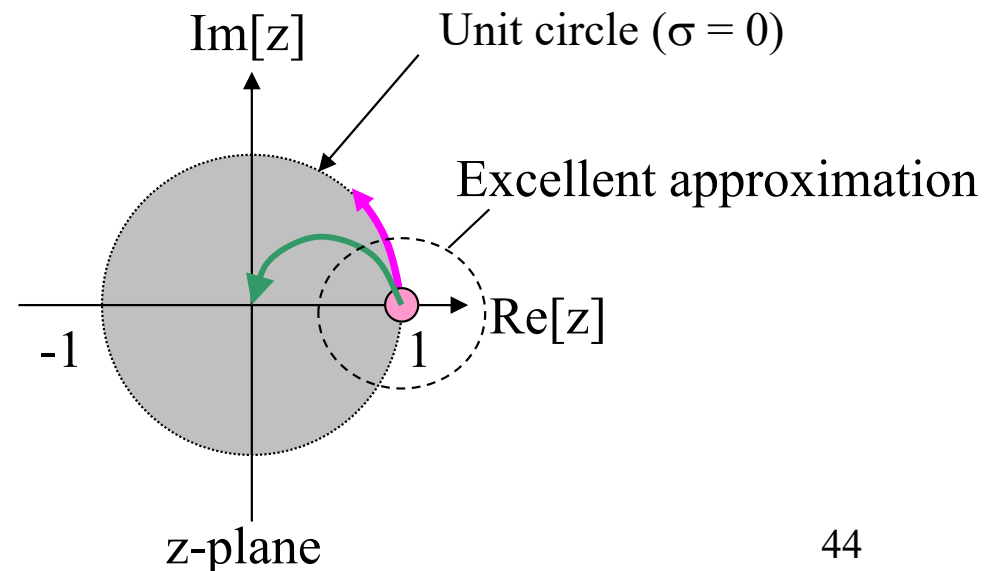
- Backward Euler Transformation (BET)

Power series expansion of  $z^{-1}$  at  $s = 0$

$$z^{-1} = e^{-sT_s} = 1 - \frac{1}{1!}(sT_s)^1 + \frac{1}{2!}(sT_s)^2 - \frac{1}{3!}(sT_s)^3 + \Lambda$$

$$\approx 1 - sT_s$$

$$\therefore s \approx \frac{1 - z^{-1}}{T_s}$$



# Approximation of Z transform 3

- Bilinear Transformation

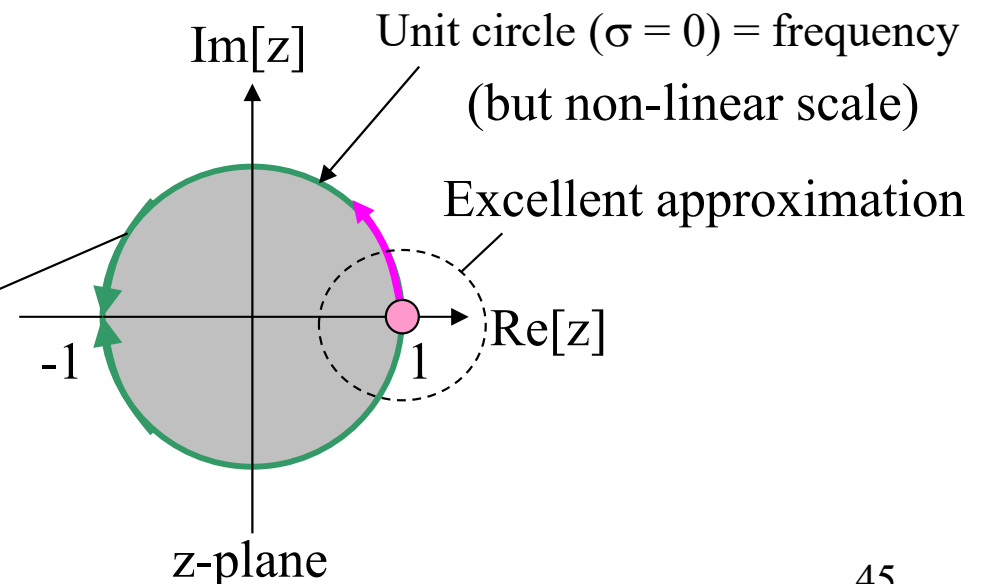
Power series expansion of  $\log_e z$  at  $s = 0$

$$sT_s = \ln z = 2 \cdot \left[ \frac{z-1}{z+1} + \frac{1}{3} \frac{(z-1)^3}{(z+1)^3} + \frac{1}{5} \frac{(z-1)^5}{(z+1)^5} + \Lambda \right]$$

$$\approx 2 \frac{z-1}{z+1}$$

$$\therefore s \approx \frac{2}{T_s} \frac{z-1}{z+1} = \frac{2}{T_s} \frac{1-z^{-1}}{1+z^{-1}}$$

$$\omega_{bilinear} = \frac{2}{T_s} \arctan \frac{\omega T_s}{2}$$



# Integration of continuous-time signal



$$v_{out}(t) = \int_0^t v_{in}(\tau) d\tau$$

↓ Laplace transform

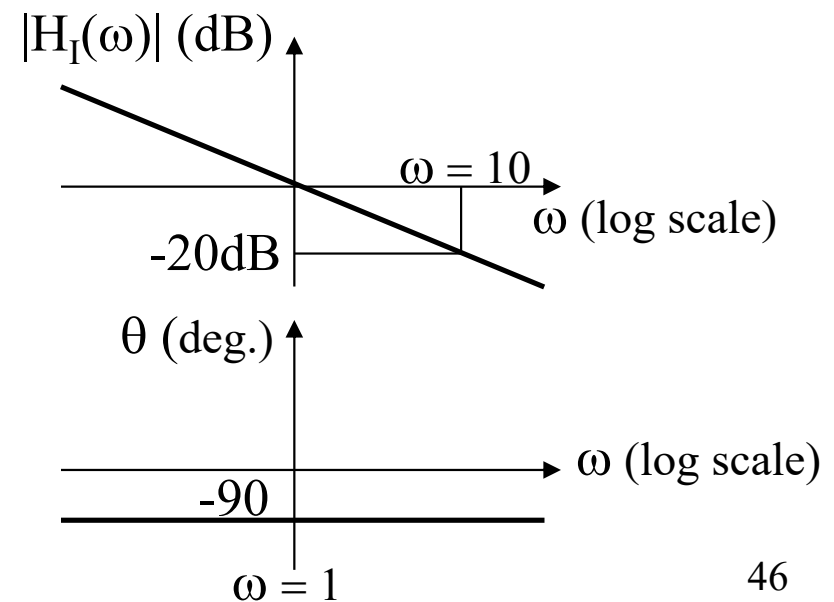
$$\begin{aligned} V_{out}(s) &= \frac{1}{s} V_{in}(s) + \frac{1}{s} \int_{-\infty}^0 v_{in}(\tau) d\tau \\ &= \frac{1}{s} V_{in}(s) \end{aligned}$$

(where the initial condition is ignored.)

Frequency domain transfer function ( $s = j\omega$ )

$$H_I(\omega) = \frac{1}{s} = \frac{1}{j\omega} = -j \frac{1}{\omega}$$

$$|H_I(\omega)| [dB] = 20 \log |H_I(\omega)| = 20 \log \frac{1}{\omega} = -20 \log \omega$$

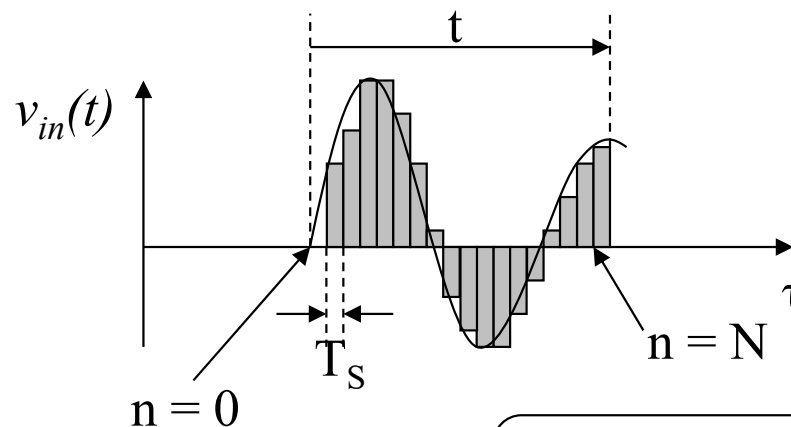




# Integration of discrete-time signal

Integration approximated with BET

$$H_I(s) = \frac{1}{s} \cong \frac{T_S}{1 - z^{-1}}$$



$$v_{out}(t) = \int_0^t v_{in}(\tau) d\tau$$

Discretized

$$v_{out}(t) = \sum_{n=0}^N v_{in}(t - nT_S) \cdot T_S$$

$\xrightarrow{Z}$

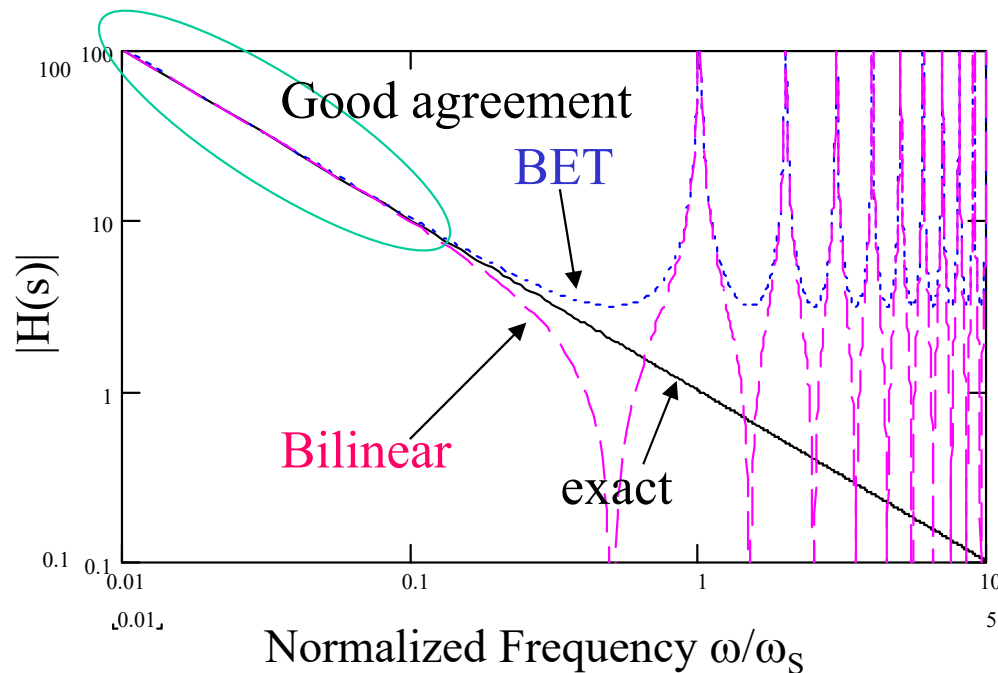
$$V_{out}(z) = T_S \sum_{n=0}^N z^{-n} V_{in}(z)$$

Geometrical series of  $z^{-1}$

$$\xrightarrow{N=\infty} \underbrace{\frac{T_S}{1 - z^{-1}}}_{H_I(s)} V_{in}(z)$$

# Integration error due to approximation

$$\left\{ \begin{array}{l} \text{Frequency domain transfer function of integrator } H(s) = \frac{1}{s} = -j \frac{1}{\omega} \\ \sigma = 0 \rightarrow z^{-1} = e^{-sT_s} = e^{-j2\pi \frac{\omega}{\omega_s}} \end{array} \right.$$



BET of Integrator

$$\frac{T_s}{1-z^{-1}} = \frac{T_s}{1-e^{-j\omega T_s}} = \frac{T_s}{1-e^{-j2\pi \frac{\omega}{\omega_s}}}$$

Bilinear transformation of integrator

$$\frac{T_s}{2} \frac{1+z^{-1}}{1-z^{-1}} = \frac{T_s}{2} \frac{1+e^{-j\omega T_s}}{1-e^{-j\omega T_s}} = \frac{T_s}{2} \frac{1+e^{-j2\pi \frac{\omega}{\omega_s}}}{1-e^{-j2\pi \frac{\omega}{\omega_s}}}$$

The approximation is excellent in  $\omega \ll \omega_s/2$ .

# Differentiation of continuous-time signal



Frequency domain transfer function ( $s = j\omega$ )

$$H_D(s) = s = j\omega$$

$$|H_D(\omega)| [dB] = 20 \log |H_D(\omega)| = 20 \log \omega$$

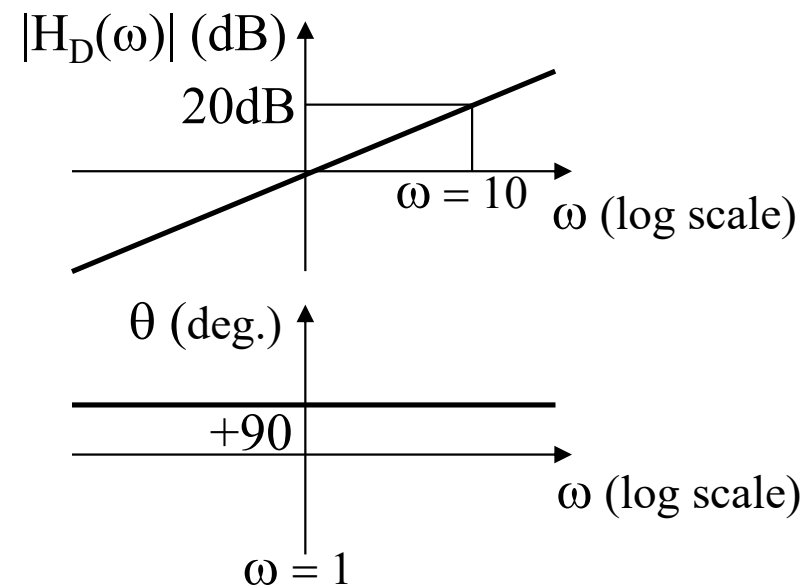
$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

↓ Laplace transform

$$V_{out}(s) = sV_{in}(s) - v_{in}(0)$$

$$V_{out}(s) = sV_{in}(s)$$

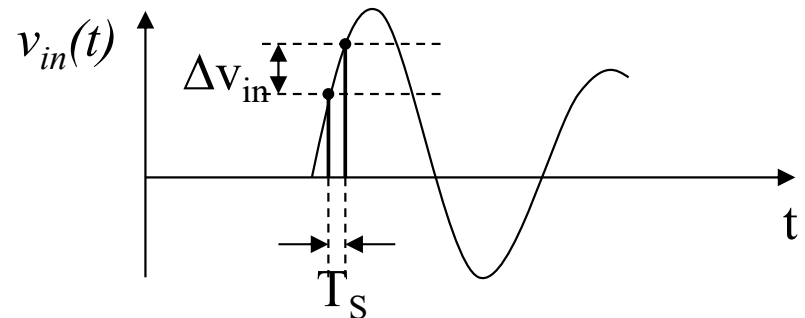
(where the initial condition is ignored.)



# Differentiation of discrete-time signal

Differentiation approximated with BET

$$H_D(s) = s \cong \frac{1 - z^{-1}}{T_S}$$



$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

Discretized

$$v_{out}(t) = \frac{v_{in}(t) - v_{in}(t - T_S)}{T_S}$$

$\Rightarrow$

$$V_{out}(z) = \frac{V_{in}(z) - z^{-1}V_{in}(z)}{T_S}$$

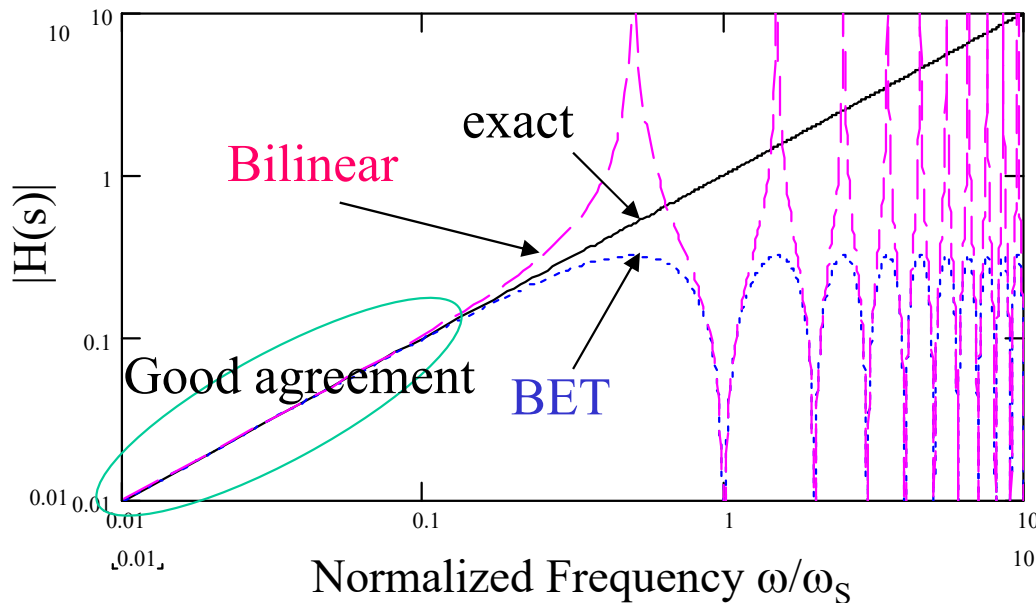
$$= \frac{1 - z^{-1}}{T_S} V_{in}(z)$$

$\underbrace{\hspace{10em}}_{H_D(s)}$

Delay element

# Differentiation error due to approximation

$$\left\{ \begin{array}{l} \text{Frequency domain transfer function of integrator } H(s) = s = j\omega \\ \sigma = 0 \rightarrow z^{-1} = e^{-sT_s} = e^{-j2\pi\frac{\omega}{\omega_s}} \end{array} \right.$$



BET of differentiator

$$\frac{1 - z^{-1}}{T_s} = \frac{1 - e^{-j\omega T_s}}{T_s} = \frac{1 - e^{-j2\pi\frac{\omega}{\omega_s}}}{T_s}$$

Bilinear transformation of differentiator

$$\frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{T_s} \frac{1 - e^{-j\omega T_s}}{1 + e^{-j\omega T_s}} = \frac{2}{T_s} \frac{1 - e^{-j2\pi\frac{\omega}{\omega_s}}}{1 + e^{-j2\pi\frac{\omega}{\omega_s}}}$$

The approximation is excellent in  $\omega \ll \omega_s/2$ .