2. Transfer function

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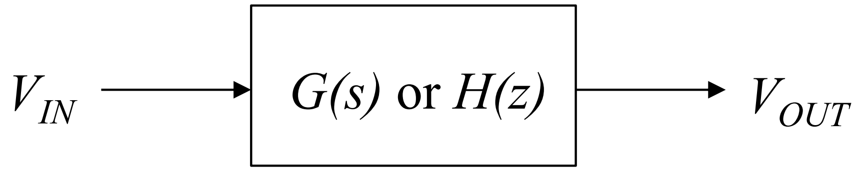
2.1 Definition of transfer function

Definition of transfer function



- Transfer function of CT circuit

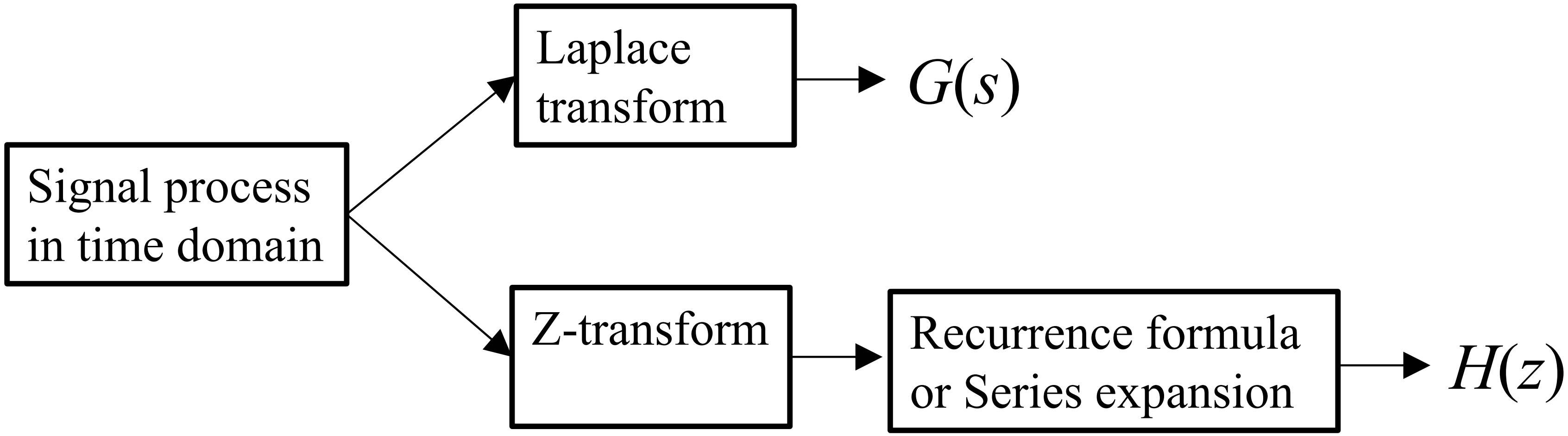
$$G(s) = \frac{V_{OUT}\,(s)}{V_{IN}\,(s)}$$

- Transfer function of DT circuit

$$H(z) = \frac{V_{OUT}\,(z)}{V_{IN}\,(z)}$$

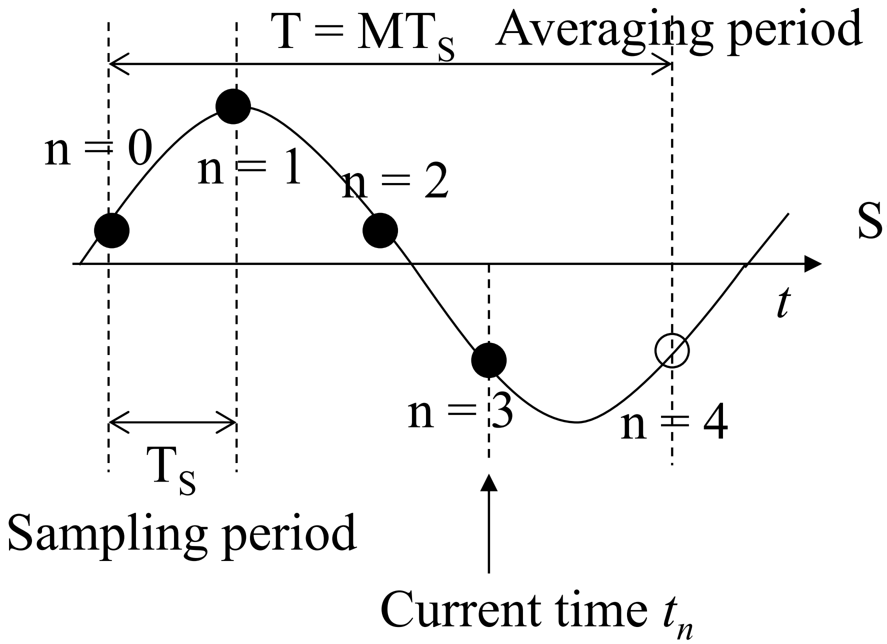
[NOTE] A transfer function is defined in s-plane or z-plane.

Method of Deriving the transfer function



Example of moving average

In the case of M = 4



Signal process in time domain

$$y(t_n) = \frac{1}{T}\sum_{n=0}^{M-1} x(t_n)T_S = \frac{1}{M}\sum_{n=0}^{M-1} x(t_n)$$

$$= \frac{1}{M}\{x(t_0) + x(t_1) + x(t_2) + x(t_3)\}$$

$$\downarrow Z$$

Series expansion of z variable

$$Y(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n}X(z)$$

$$= \frac{1}{M}\left(1 + z^{-1} + \cdots + z^{-(M-1)}\right)X(z)$$

$$= \frac{1}{M}\left(z^{-3} + z^{-2} + z^{-1} + z^{0}\right)X(z)$$

$$\downarrow$$

Transfer function

$$MY(z) = (z^{-1} + z^{-2} + \cdots + z^{-M})X(z) + (z^0 - z^{-M})X(z) = z^{-1}MY(z) + (1 - z^{-M})X(z)$$

$$(1 - z^{-1})MY(z) = (1 - z^{-M})X(z) \longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}\frac{(1 - z^{-M})}{(1 - z^{-1})}$$

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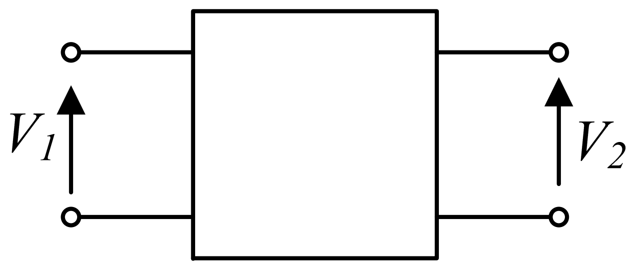
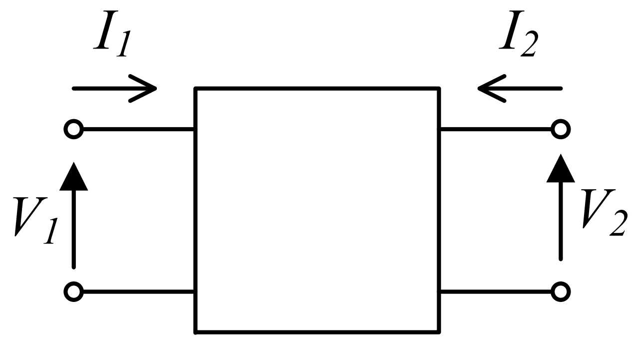
Two-port network parameters and transfer function



Transfer function



Two-port network parameters

$$V_2(s) = H(s) \cdot V_1(s)$$

The transfer function varies in according with the load impedance.

$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ I_1(j\omega) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(j\omega) \\ -I_2(j\omega) \end{bmatrix}$$

Two-port network parameters do not vary in according with the load impedance.

スライド 6

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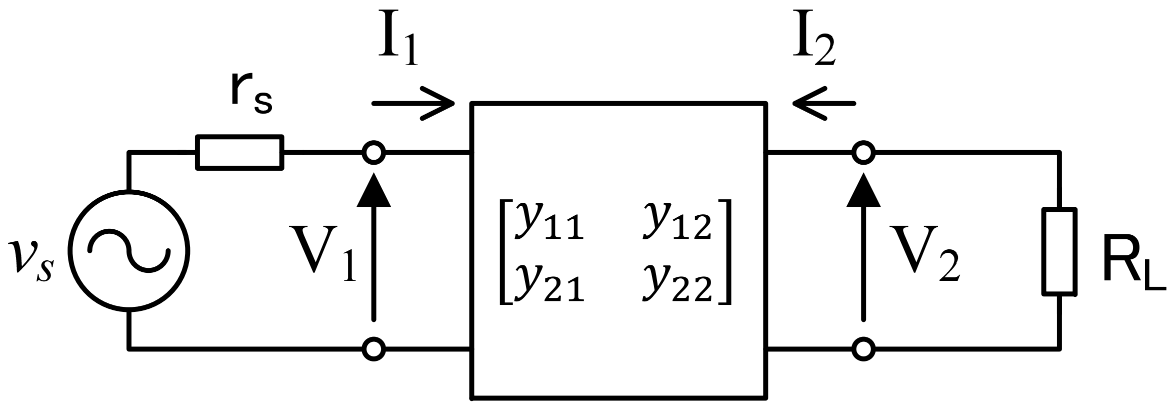
Load-dependence of transfer function



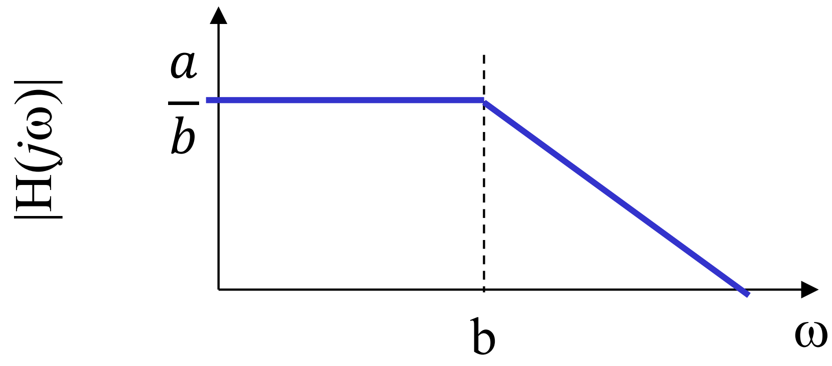
$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} \Rightarrow \begin{cases} I_2 = y_{21}V_1 + y_{22}V_2 \\ -V_2 = R_L I_2 \end{cases}$$

$$H(s = j\omega) = \frac{V_2}{V_1} = -\frac{y_{21}}{y_{22} + \frac{1}{R_L}}$$

[Note] Frequency domain transfer function

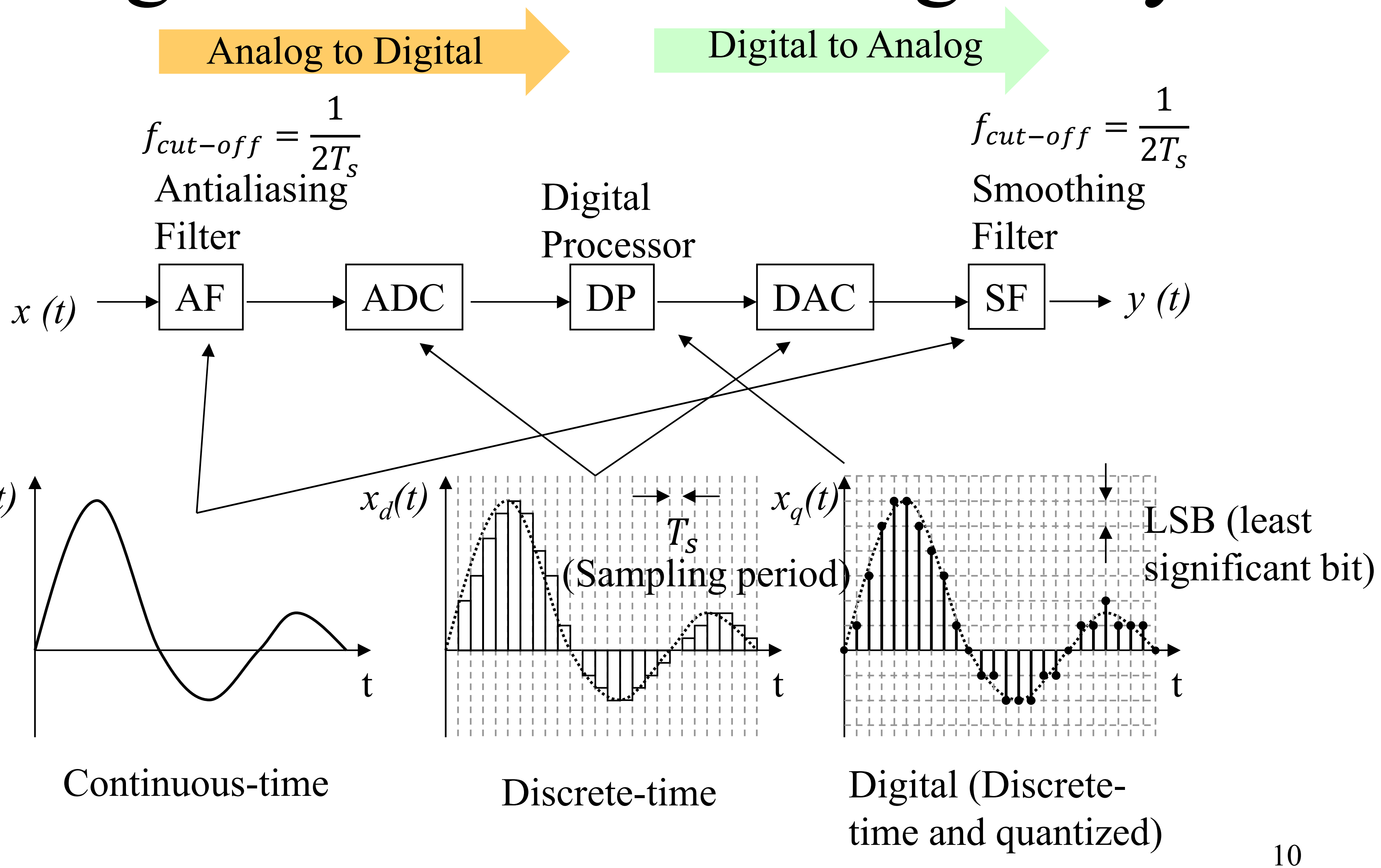
- The frequency domain transfer function (or the frequency response) is obtained from the transfer function under the condition that $s = j\omega$. (However, the reverse is not always true.)
- The frequency transfer function means the steady state characteristic of the circuit which is stimulated with $e^{j\omega t}$.

$$H(s) = \frac{a}{b+s} \xrightarrow{s=j\omega} H(j\omega) = \frac{a}{b+j\omega} = \frac{\frac{a}{b}}{1+j\frac{\omega}{b}}$$

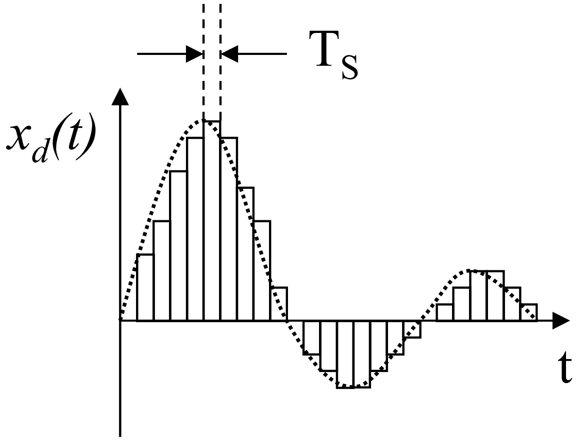


2.2 Waveforms in mixed-signal circuits

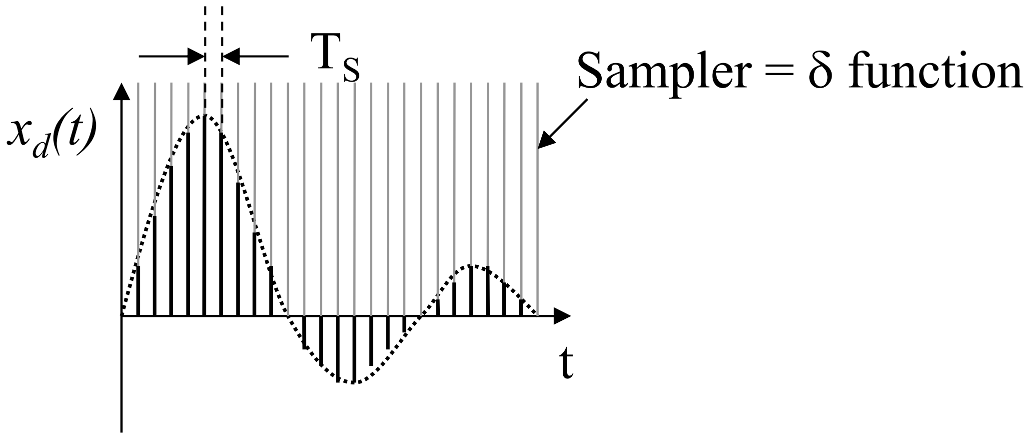
Configuration of mixed-signal system



Expressions of discrete-time signal



Step sampling
(Zero-order hold)



Impulse sampling
(Pulse Amplitude Modulation or PAM)

$$x_{du}(t) = \sum_n x(nT_S) \cdot \underbrace{\{u(t - nT_S) - u(t - (n+1)T_S)\}}_{\text{Step Sampler}}$$

$$x_d(t) = \sum_n x(t) \cdot \underbrace{\delta(t - nT_S)}_{\text{Impulse Sampler}}$$

NOTE: The discrete analog signal is practically obtained by S/H, but the signal can be handled similar to the impulse sequences (See slide 11).

Laplace transform and Z transform

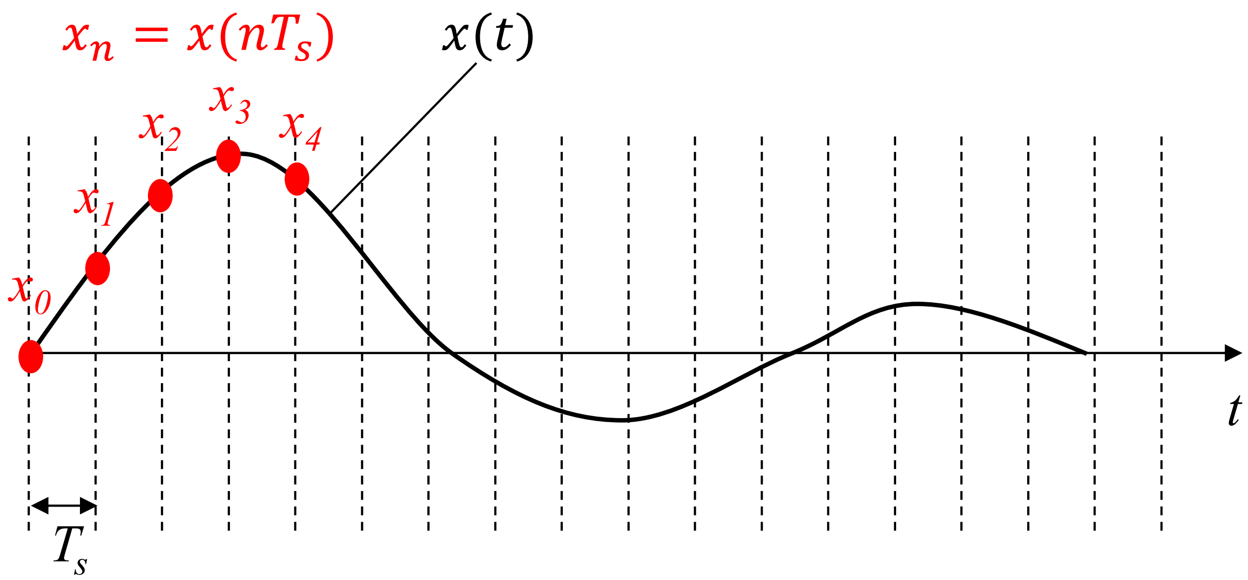
Continuous-time signal $x(t)$ $\xrightarrow{\mathcal{L}}$ $X(s) = \int_0^{\infty} x(t)e^{-st}dt$

Discrete-time signal $x(t)$ $\xrightarrow{Z}$ $X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

$$s = \sigma + j\omega$$

$$\downarrow$$

$$z^{-1} = e^{-sT_s}$$



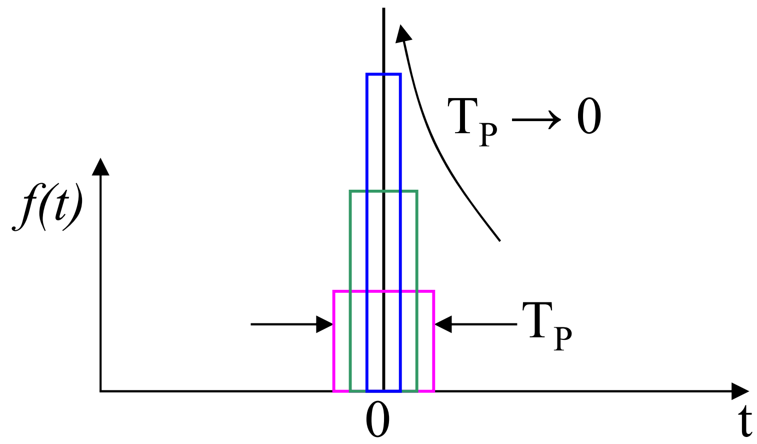
Characteristic of δ function

Step function f

$$f(t) = \frac{1}{T_P}\{u(t + \frac{T_P}{2}) - u(t - \frac{T_P}{2})\}$$

$$\int_{-\infty}^{\infty} f(t)dt = 1 \qquad (1)$$

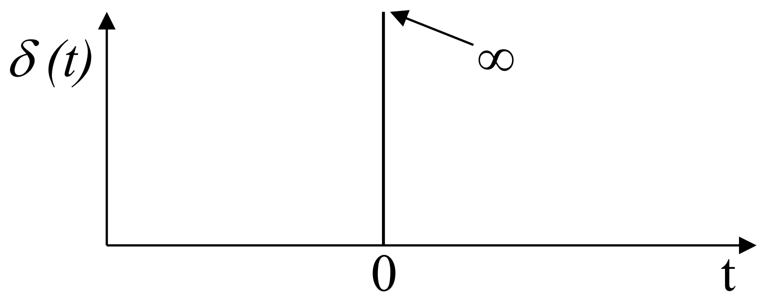
$$\delta(t) \equiv \lim_{T_P \to 0} f(t) \qquad (2)$$



Delta function δ

Important

$$\begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 & (3) \\ \delta(t) = 0 \quad (t \neq 0) & (4) \end{cases}$$



Z transform of discrete-time signal

PAM signal

$$x_d(t) = \sum_n x(t)\delta(t - nT_s)$$

⇓ Laplace transform

$$X_d(s) = \int_0^\infty x_d(t)e^{-st}dt$$

$$= \int_0^\infty \sum_n x(t)\delta(t - nT_s)e^{-st}dt$$

$$= \sum_n \int_0^\infty x(t)\delta(t - nT_s)e^{-st}dt$$

$$= \sum_n x(nT_s)e^{-s(nT_s)}$$

$$z^{-1} = e^{-sT_s}$$

$$= \sum_n x(nT_s)z^{-n}$$

Zero-order hold signal

$$x_u(t) = \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}$$

⇓ Laplace transform

$$X_u(s) = \int_0^\infty x_u(t)e^{-st}dt$$

$$= \int_0^\infty \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s) \int_0^\infty \{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s) \frac{e^{-snT_s}(1 - e^{-sT_s})}{s}$$

$$z^{-1} = e^{-sT_s}$$

$$= \underbrace{\frac{1 - e^{-sT_s}}{s}}_{\text{Transfer function of step sampler}} \underbrace{\sum_n x(nT_s)z^{-n}}_{\text{PAM signal}}$$

Z transform of PAM signal

Continuous-time signal $\quad x(t) \xrightarrow{\text{Laplace transform}} X(s) \xrightarrow{z = e^{sT_s}} X(z)$

$\downarrow$ Impulse sampling

Discrete-time signal

$$x_d(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

$\downarrow$ Z transform

Z transform using delay z^{-n}

$$X_d(z) = \sum_n x_n z^{-n} \qquad (5)$$

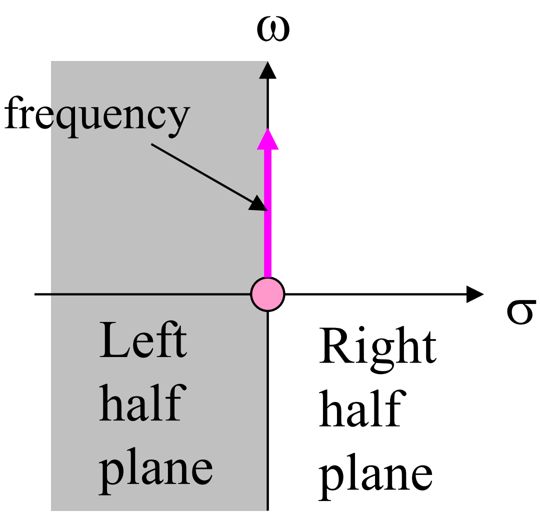
A discrete-time signal can be easily transformed by using Eq.(5).
You do not need to calculate by Laplace transform.

s-plane and z-plane

s-plane | Time domain | z-plane

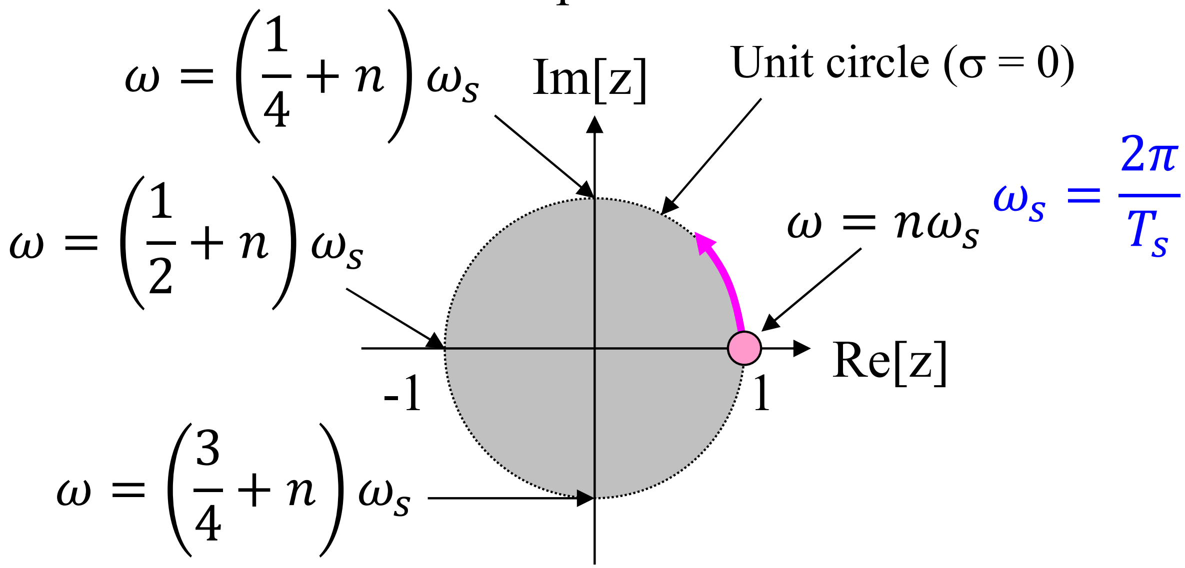
$$X_d(s) = \sum_{n} x(nT_s)e^{-s(nT_s)} \longleftarrow x_d(t) = \sum_{n=0}^{\infty} x(t)\delta(t - nT_s) \longrightarrow X_d(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

s-plane



$$s = \sigma + j\omega$$

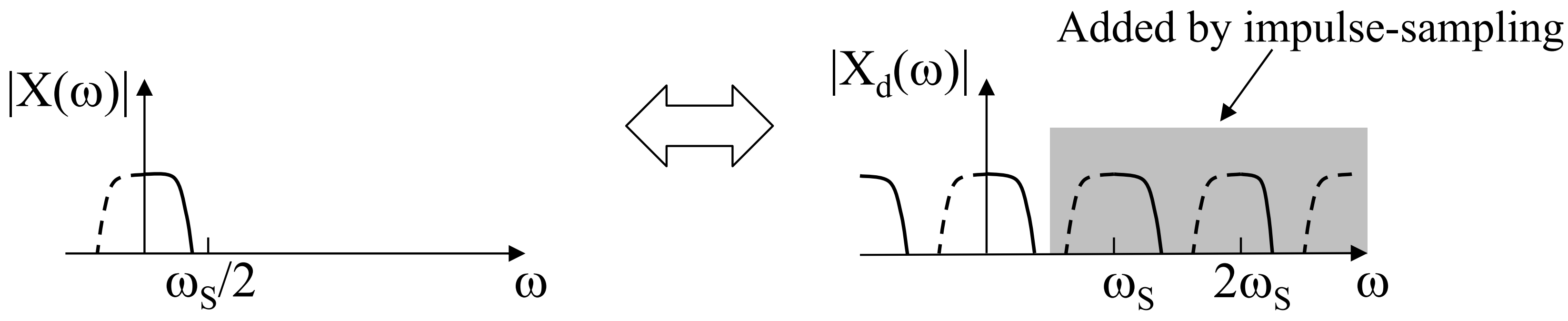
z-plane



$$z = e^{sT_s} = e^{\sigma T_s}(\cos \omega T_s + j \sin \omega T_s)$$

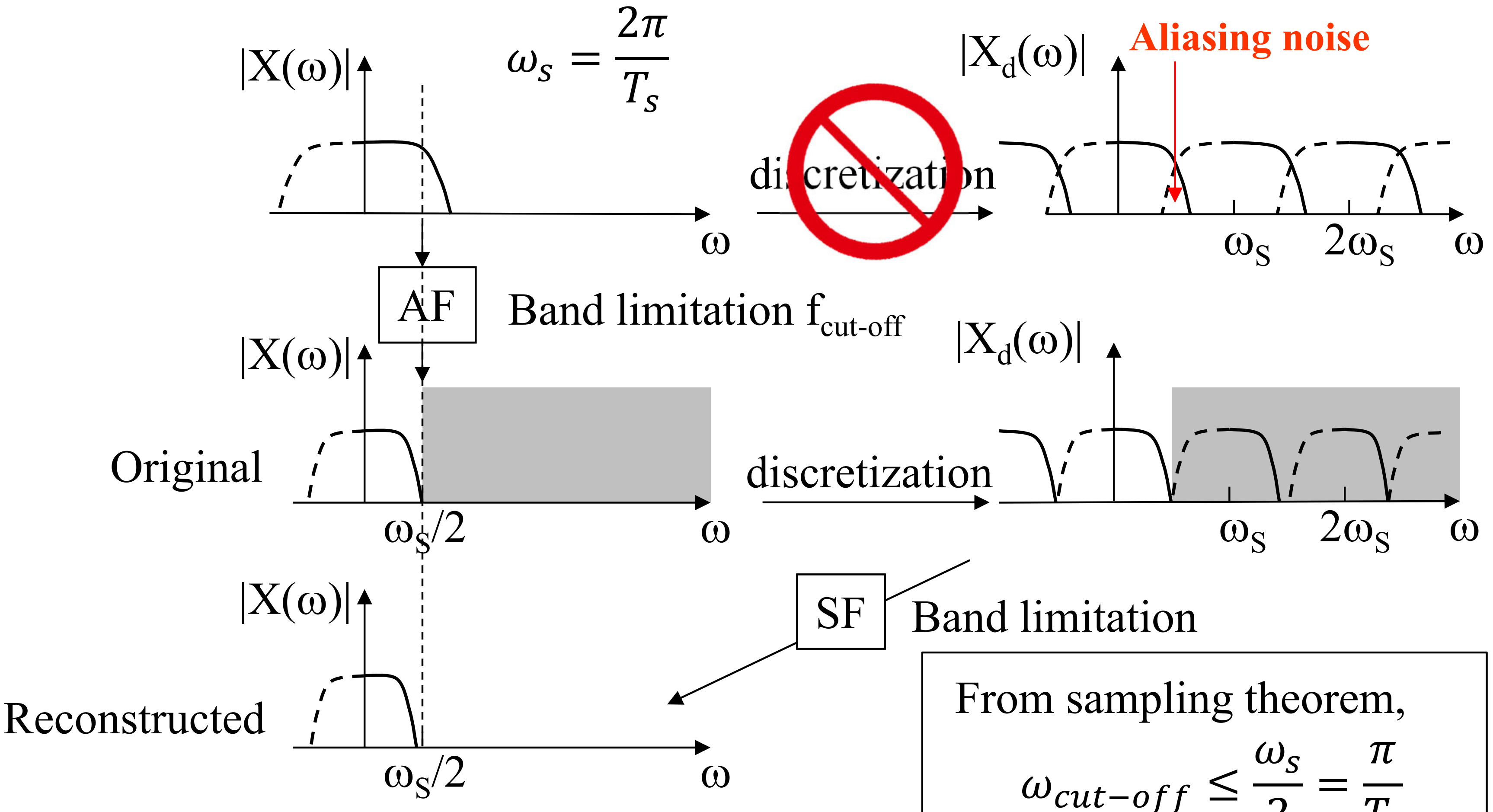
Spectrum of PAM signal

Spectrum of continuous-time signal   Spectrum of discrete-time signal



See Appendix 2(1) - 2(3).

Antialiasing and smoothing of signals

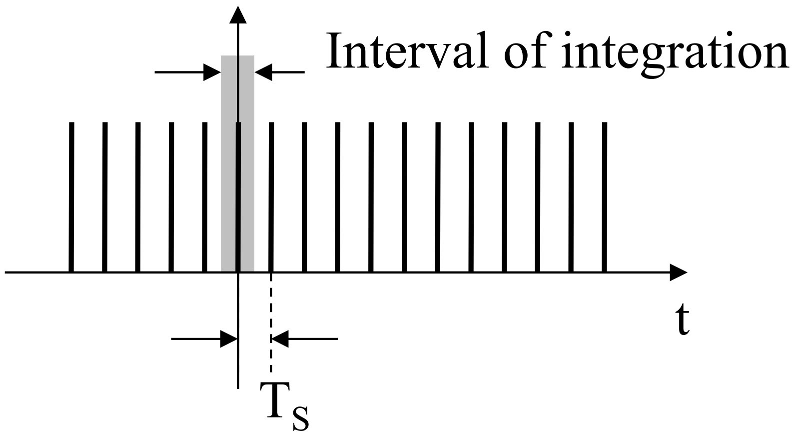


From sampling theorem,

$$\omega_{cut-off} \leq \frac{\omega_s}{2} = \frac{\pi}{T_s}$$

NOTE: AF and SF can be implemented in continuous-time circuits.

Appendix 2(1) Fourier series of impulse sampler



$$\delta_T(t) = \sum_n \delta(t - nT_S)$$

$$\delta_T(t) = \sum_n c_n e^{jn\frac{2\pi}{T_S}t}$$

$$c_n = \frac{1}{T_S}\int_{-\frac{T_S}{2}}^{\frac{T_S}{2}} \delta_T(t)\cdot e^{jn\frac{2\pi}{T_S}t}\,dt$$

$$= \frac{1}{T_S} e^{jn\frac{2\pi}{T_S}0} = \frac{1}{T_S}$$

$$\therefore \delta_T(t) = \frac{1}{T_S}\sum_n e^{jn\frac{2\pi}{T_S}t}$$

Appendix 2(2) Spectrum of PAM

$$x_d(t) = \sum_n x(t)\cdot\delta(t - nT_S) = x(t)\cdot\delta_T(t) = \frac{1}{T_S}\sum_n x(t)\cdot e^{jn\frac{2\pi}{T_S}t}$$

Translation Theorem $\mathcal{L}[e^{-at}f(t)] = F(s+a)$

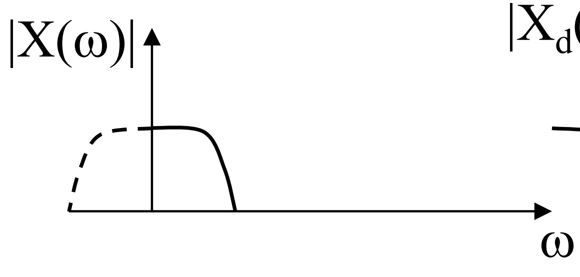
$$X_d(s) = \frac{1}{T_S}\sum_n X(s - jn\frac{2\pi}{T_S}) = \frac{1}{T_S}\sum_n X(s - jn\omega_S)$$

$$\omega_S = \frac{2\pi}{T_S}$$

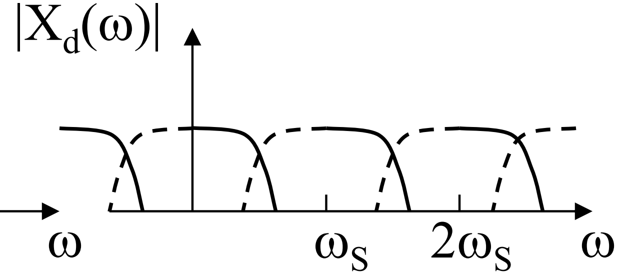
Let's say, $s = j\omega$

$$X_d(\omega) = \frac{1}{T_S}\sum_n X\{j(\omega - n\omega_S)\}$$

Continuous-time signal



PAM signal

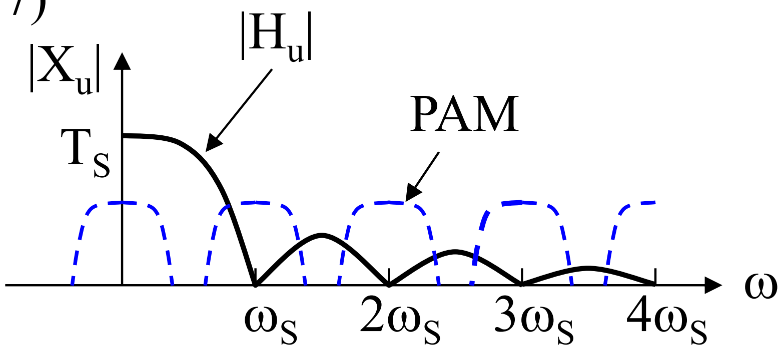


Appendix 2(3) Spectrum of zero-order hold

Laplace transform of zero-order hold signal (Slide 7)

$$X_u(s) = \boxed{\frac{1 - s^{-sT_s}}{s}} \underbrace{\sum_n x(nT_s) z^{-n}}_{\text{PAM signal}}$$

Transfer function of step sampler $H_u(s)$



$$H_u(s) = \frac{1 - e^{-sT_S}}{s} = e^{-\frac{sT_S}{2}} \frac{e^{\frac{sT_S}{2}} - e^{-\frac{sT_S}{2}}}{s} \xrightarrow{s=j\omega} e^{-j\frac{\omega T_S}{2}} \frac{e^{j\frac{\omega T_S}{2}} - e^{-j\frac{\omega T_S}{2}}}{j\omega} = e^{-j\frac{\omega T_S}{2}} T_S \frac{\sin\frac{\omega T_S}{2}}{\frac{\omega T_S}{2}}$$

$$= \underbrace{e^{-j\frac{\omega T_S}{2}}}_{\text{Phase}} \underbrace{T_S \mathrm{Sinc}\left(\frac{\omega T_S}{2}\right)}_{\text{Amplitude}}$$

[NOTE] The spectrum of zero-order hold signal is deviated from the spectrum of PAM signal by the step sampler $H_u(s)$. Therefore, the smoothing filter after DAC must has the Sinc^{-1} characteristic.

2.3 Transfer function of continuous-time analog circuits

Integration and Differentiation

Linear circuit equation in time domain

Differentia equations in time domain

Integral operation   Differential operation

s^{-1} s

Linear circuit equation of s-variable

Polynomial expressions of variable s

The transfer function is solvable with arithmetic operations .

Transfer function $H(s) = \dfrac{Output\ signal\ (s)}{Input\ signal\ (s)}$

Definition of transfer function

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

- Transfer function
  - $s = \sigma + j\omega$: Transfer function
  - $s = j\omega$: Frequency domain transfer function
- Pole and Zero
  - $1/H(s_p) = 0$, for the complex number s_p at a location of pole in s-plane
  - $H(s_z) = 0$, for the complex number s_z at a location of zero in s-plane
- Corner frequency of pole and zero
  - A corner frequency in Bode diagram is observed as a consequence of pole and zero.
  - Pole frequency: The corner of amplitude response is convex downward.
  - Zero frequency: The corner of amplitude response is convex upward.

Decibel (dB)

The vertical axis of Bode diagrams is plotted in the decibel scale (dB).The decibel indicates the absolute value ratio of the signal amplitude.

Decibel of voltage and current signal $dB = 20log_{10}\left|\frac{V2}{V1}\right| = 20log_{10}|H(\omega)|$

Decibel of signal power $dB = 10log_{10}\left|\frac{P2}{P1}\right|$

Note: dBm is not ratio, but the absolute value of the signal power in mW.

$$dBm = 10log_{10}P(mW)$$

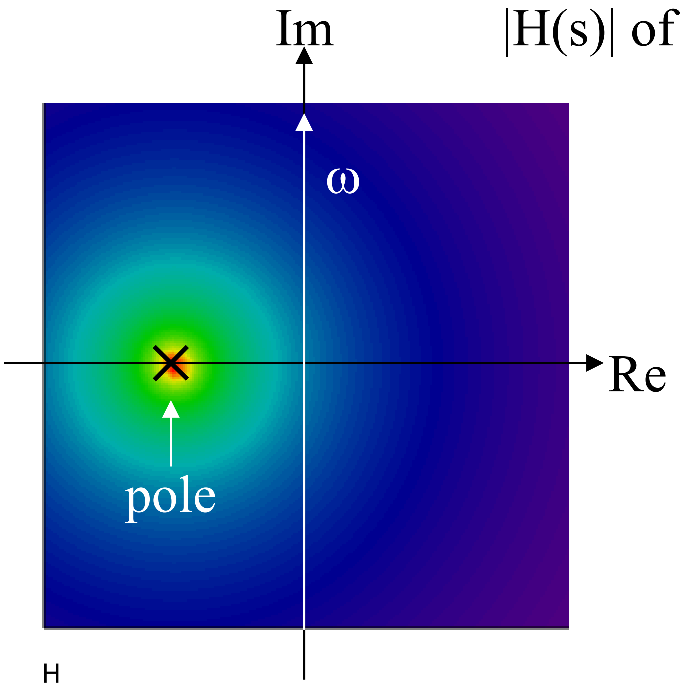
1-pole transfer function

$$H(s)=\frac{a\cdot s+b}{s+c}=\frac{b}{c}\frac{(1+\frac{as}{b})}{(1+\frac{s}{c})}$$

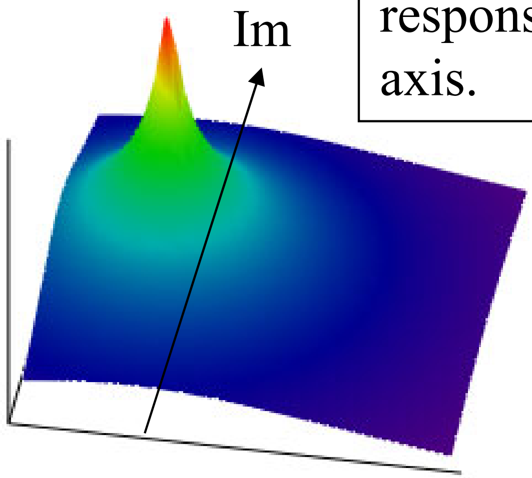
(a, b, c = real number)

Type of frequency response

$a=0,\quad b\neq 0$	LPF
$a\neq 0,\quad b=0$	HPF



$$H(s)=\frac{b}{s+c}$$

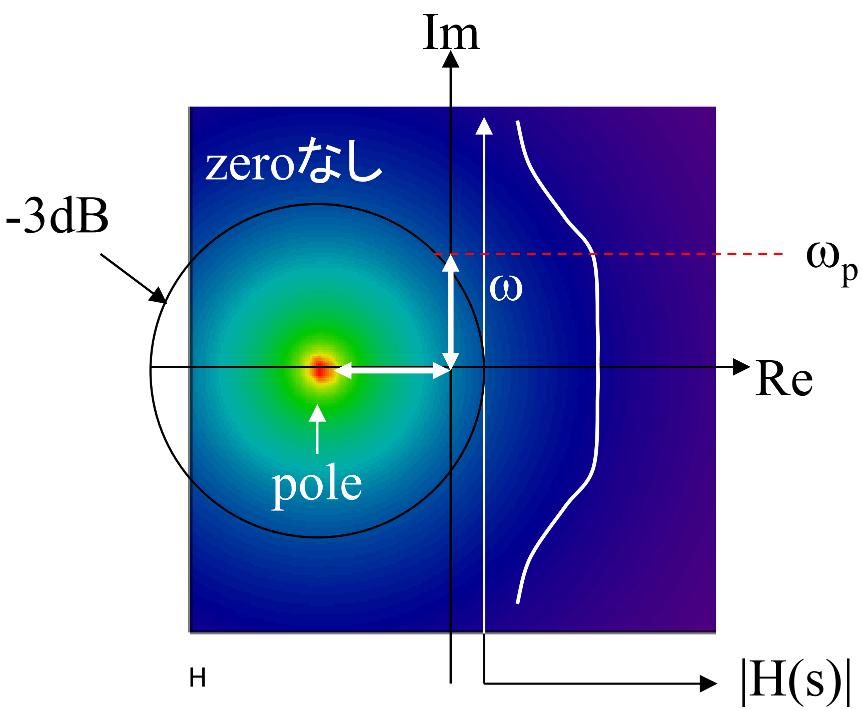


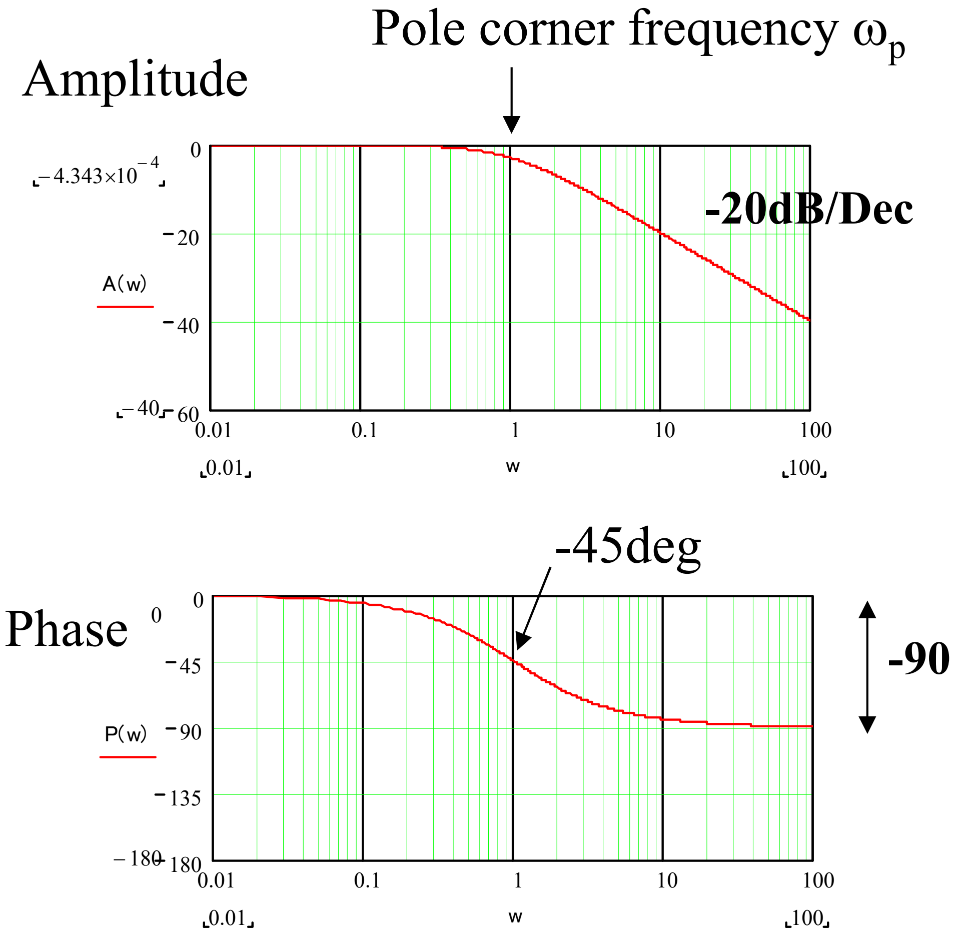
The Bode diagram is represents the amplitude response in the imaginary axis.

Bode diagram of 1st order LPF

Transfer function

$$H(s) = \frac{b}{s + c}$$



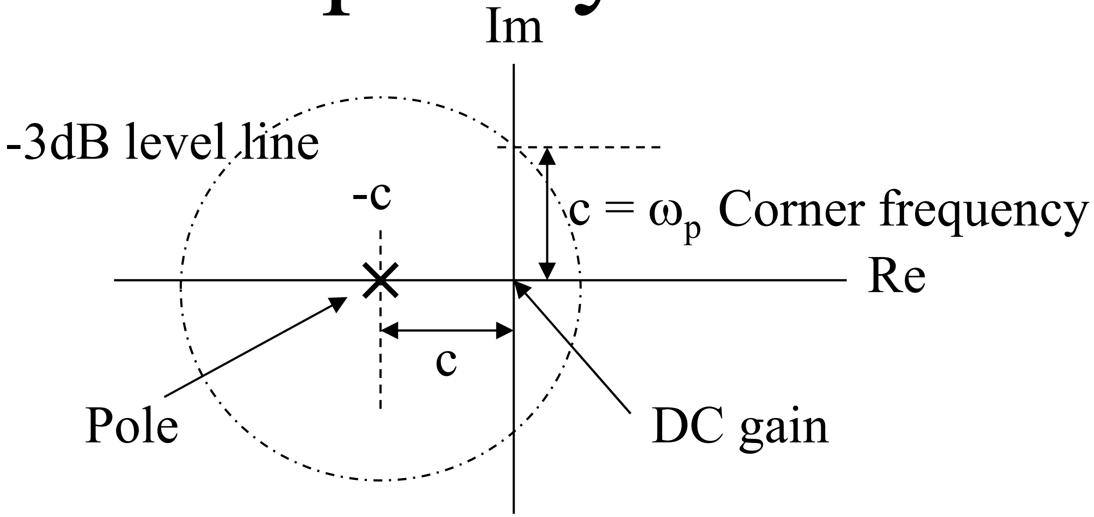


Positional relation between pole and corner frequency

$$H(s) = \frac{b}{s+c}$$

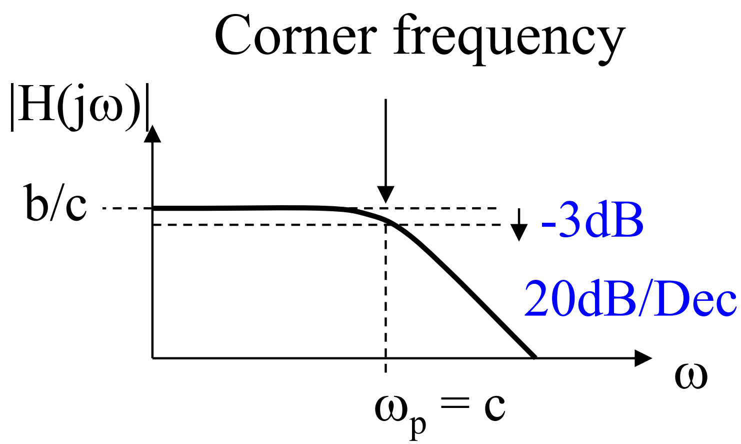
$$H(j\omega) = \frac{\frac{b}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\frac{b}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \frac{b}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}}\frac{b}{c} \\ \omega >> c \rightarrow |H(j\omega)| = \frac{b}{\omega} \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



1-pole, 1-zero transfer function

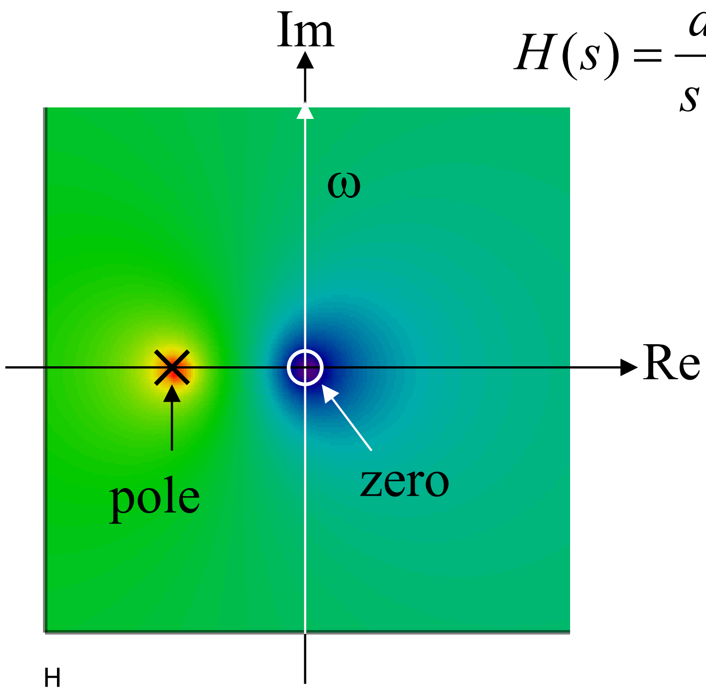
$$H(s) = \frac{a \cdot s + b}{s + c}$$ (a, b, c = real number)

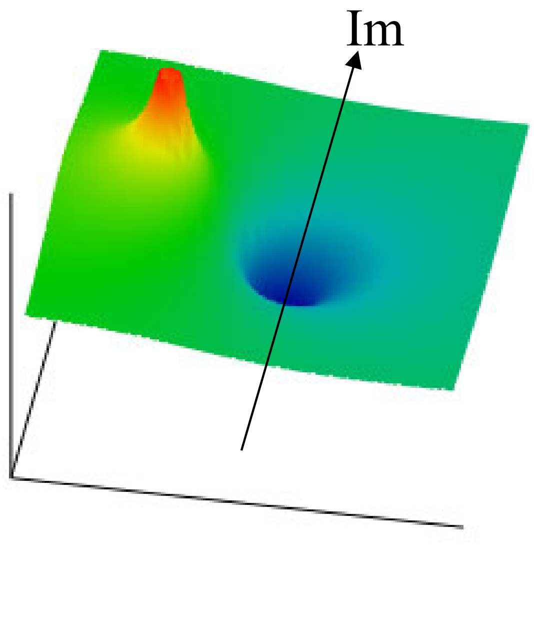
Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF

|H(s)| of HPF

$$H(s) = \frac{a \cdot s}{s + c}$$

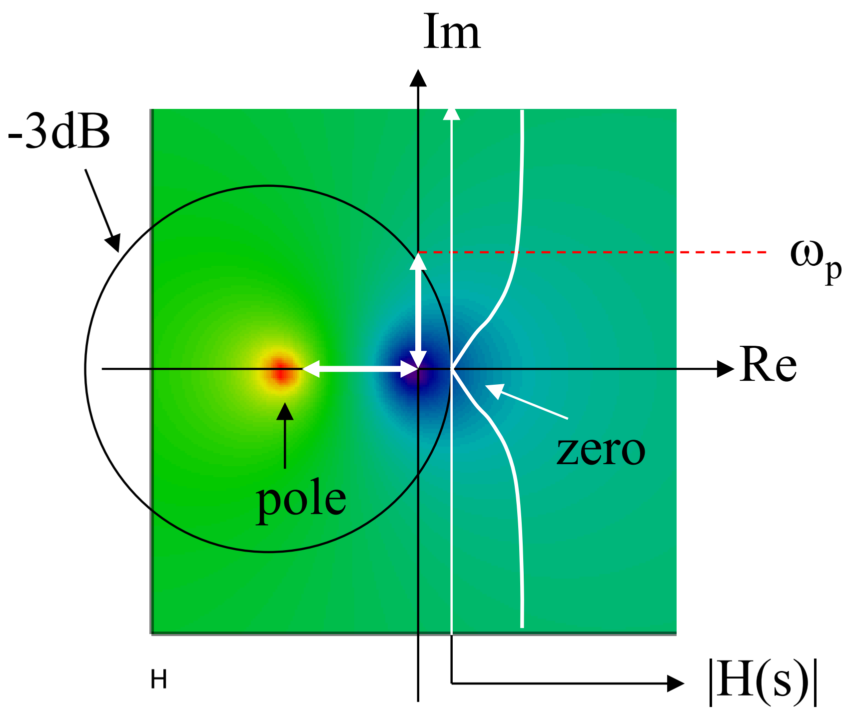


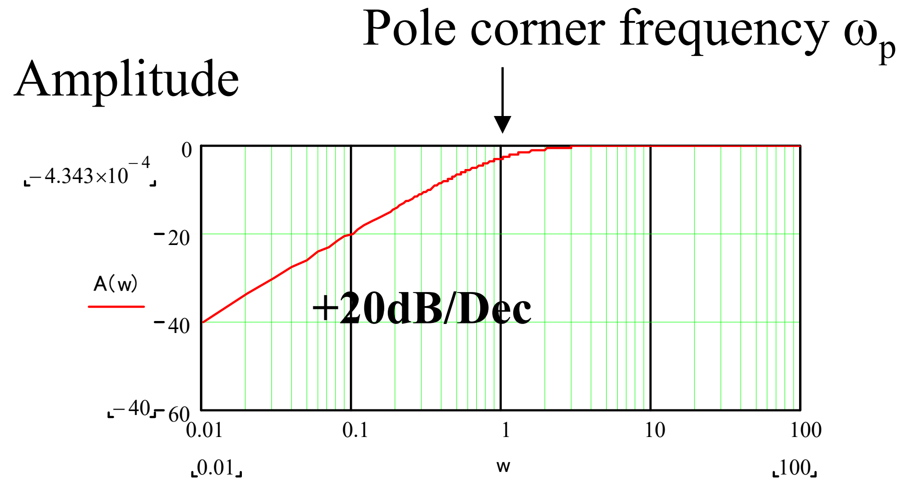


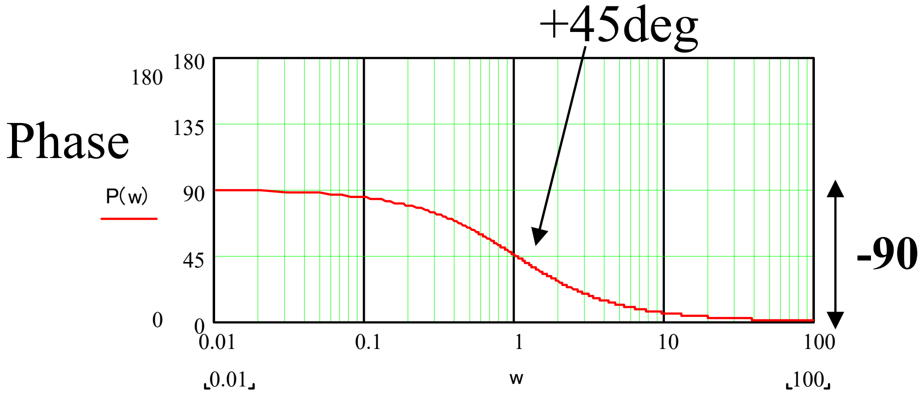
Bode diagram of 1st order HPF

Transfer function

$$H(s) = \frac{a \cdot s}{s + c}$$





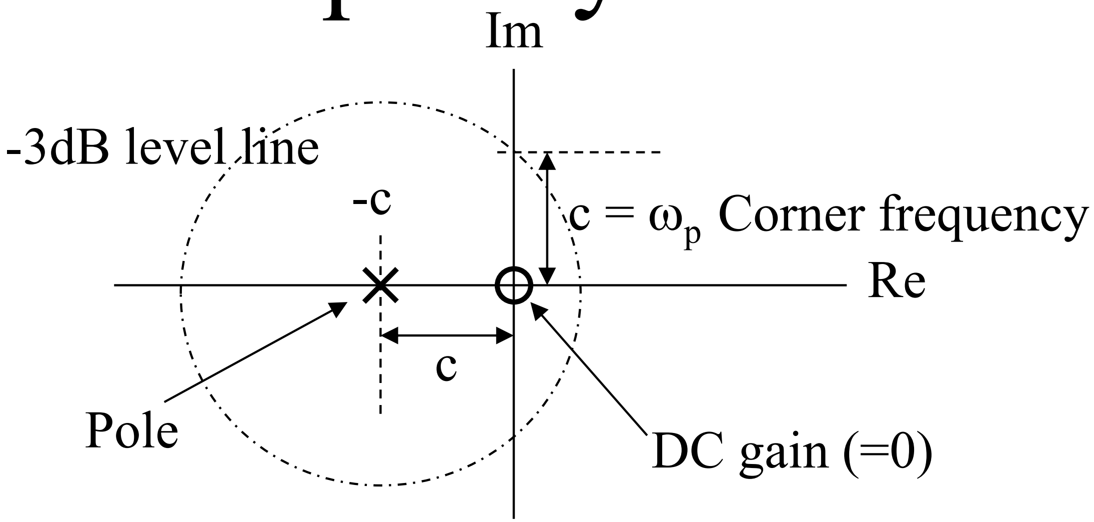


Positional relation between pole and corner frequency

$$H(s) = \frac{as}{s+c}$$

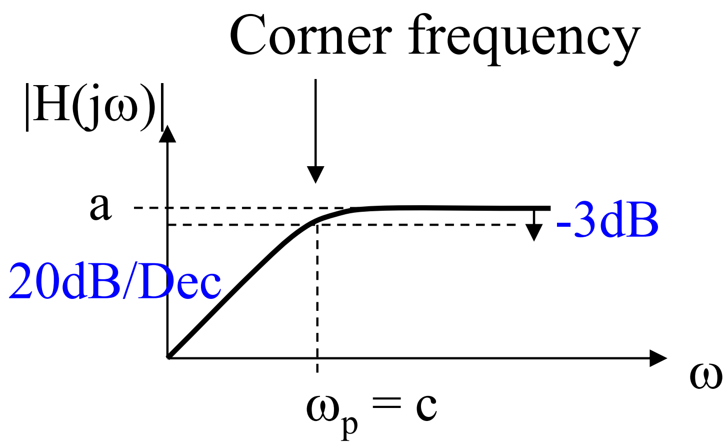
$$H(j\omega) = \frac{j\omega \frac{a}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\omega \frac{a}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \omega \frac{a}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{a}{\sqrt{2}} \\ \omega >> c \rightarrow |H(j\omega)| = a \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



2-pole transfer function

$$H(s)=\frac{a\cdot s^2+b\cdot s+c}{s^2+d\cdot s+e}$$

(a, b, c = real number)

Type of frequency response

$a=b=0, \quad c\neq 0$	LPF
$a=c=0, \quad b\neq 0$	BPF
$b=c=0, \quad a\neq 0$	HPF
$b=0, \quad a\neq 0, \quad c\neq 0$	BEF

|H(s)| of LPF

$$H(s)=\frac{c}{s^2+d\cdot s+e}$$

Note: If the denominator is factorable, the transfer function has 2 real poles.

Solution of the pole

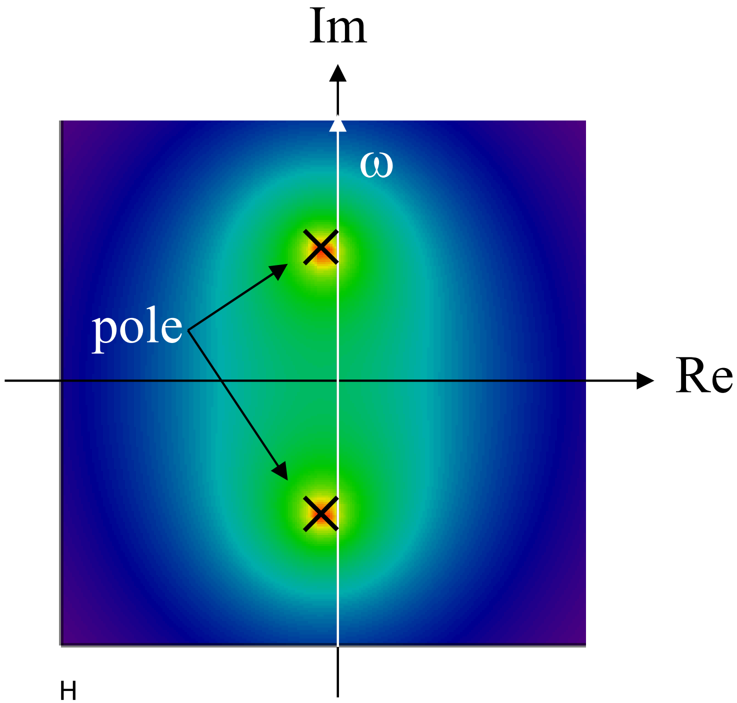
$$D(s)=s^2+d\cdot s+e=0$$

$$s=-\frac{d}{2}\pm j\sqrt{e-\frac{d^2}{4}}$$

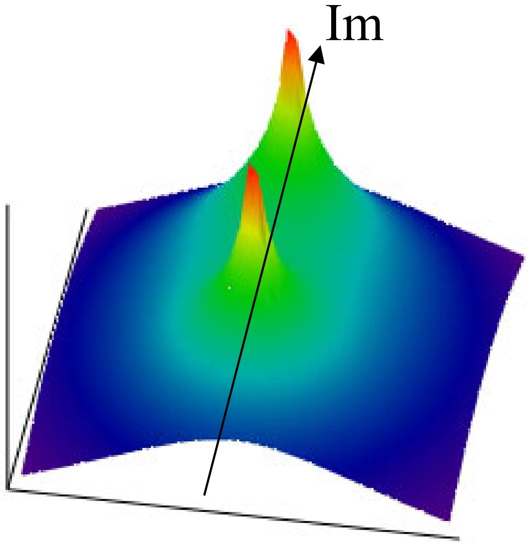
Complex number of 2 poles

Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$ の場合



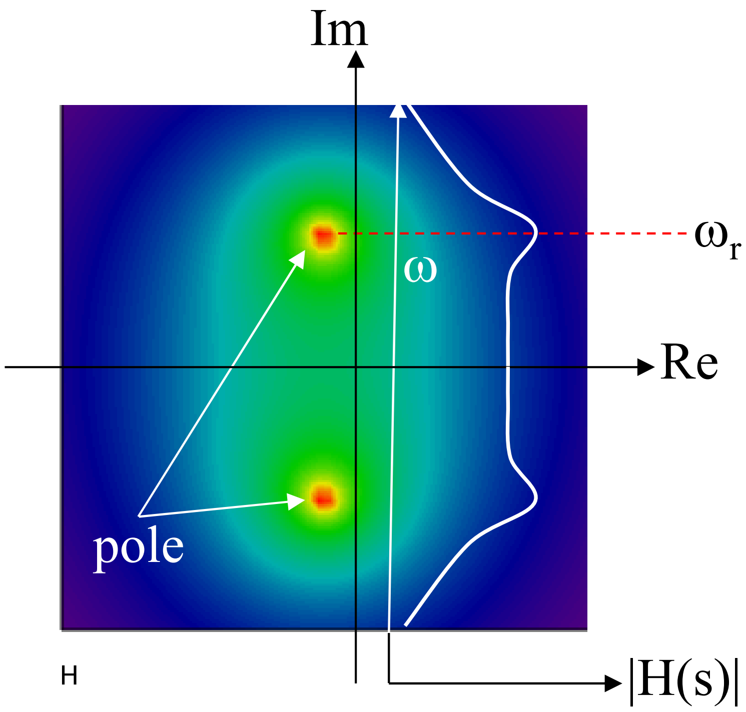
H

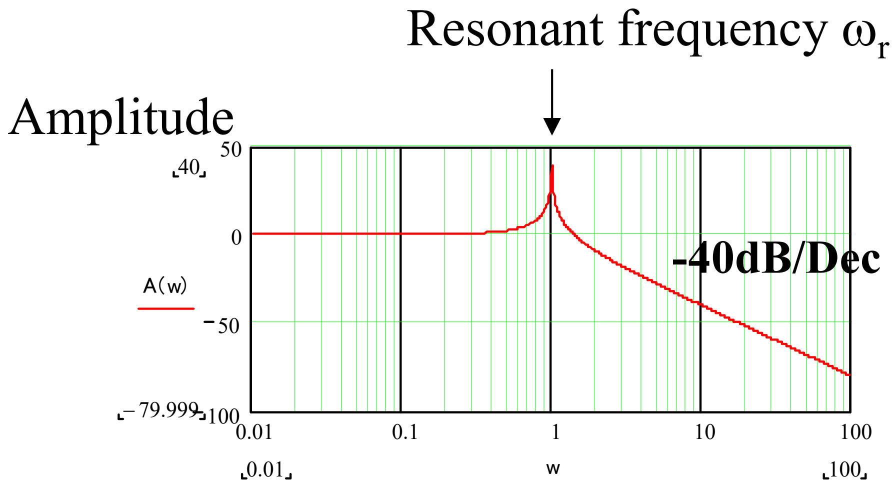


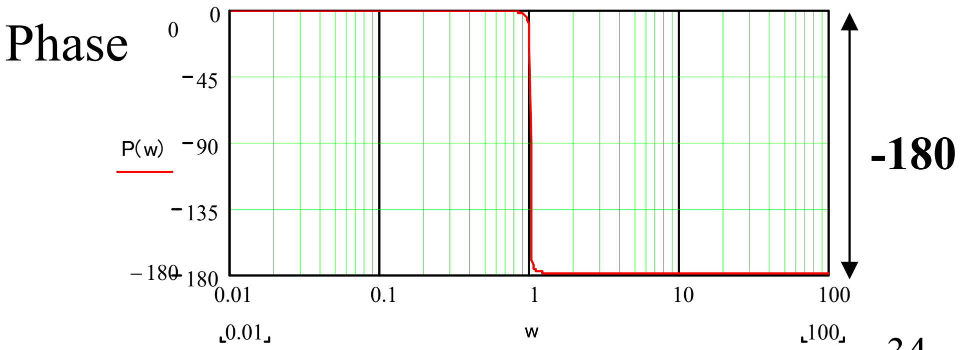
H

Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$





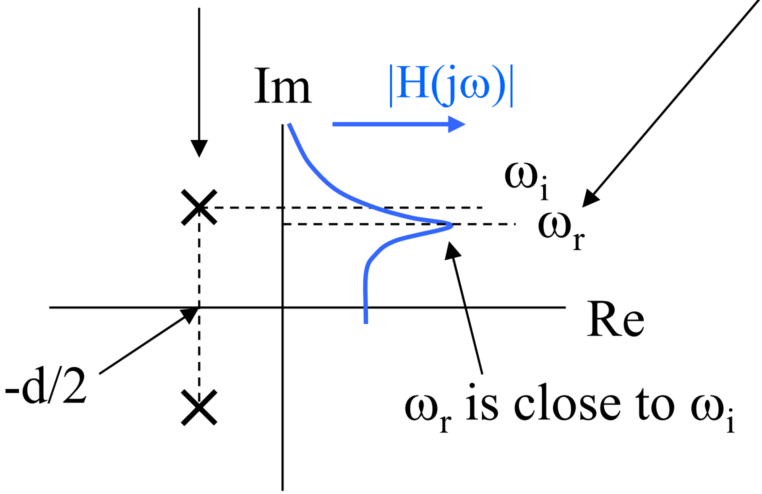


Positional relation between pole and corner frequency

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

$s^2 + d \cdot s + e = 0$ のとき

$$s = \underbrace{-\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}}}_{\text{Location of the pole in s-plane}} \equiv -\frac{d}{2} \pm j\omega_i$$



$$|H(j\omega)| = \frac{c}{\sqrt{\{(j\omega)^2 + d(j\omega) + e\}\{(-j\omega)^2 + d(-j\omega) + e\}}}$$

$$= \frac{c}{\sqrt{\{\omega^2 - (e - \frac{d^2}{2})\}^2 + d^2(e - \frac{d^2}{4})}}$$

$$\omega_r \equiv \sqrt{(e - \frac{d^2}{2})}$$

$$|H(j\omega)| = \frac{c}{\sqrt{\{\omega^2 - \omega_r^2\}^2 + d^2(e - \frac{d^2}{4})}}$$

If $\omega = \omega_r$, the amplitude reach a maximum.

$$|H(j\omega_r)| = \frac{c}{d\sqrt{e - \frac{d^2}{4}}}$$

The smaller d causes the higher peak.

2.4 Transfer function of discrete-time analog/digital circuits

Definition of transfer function on z-plane

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

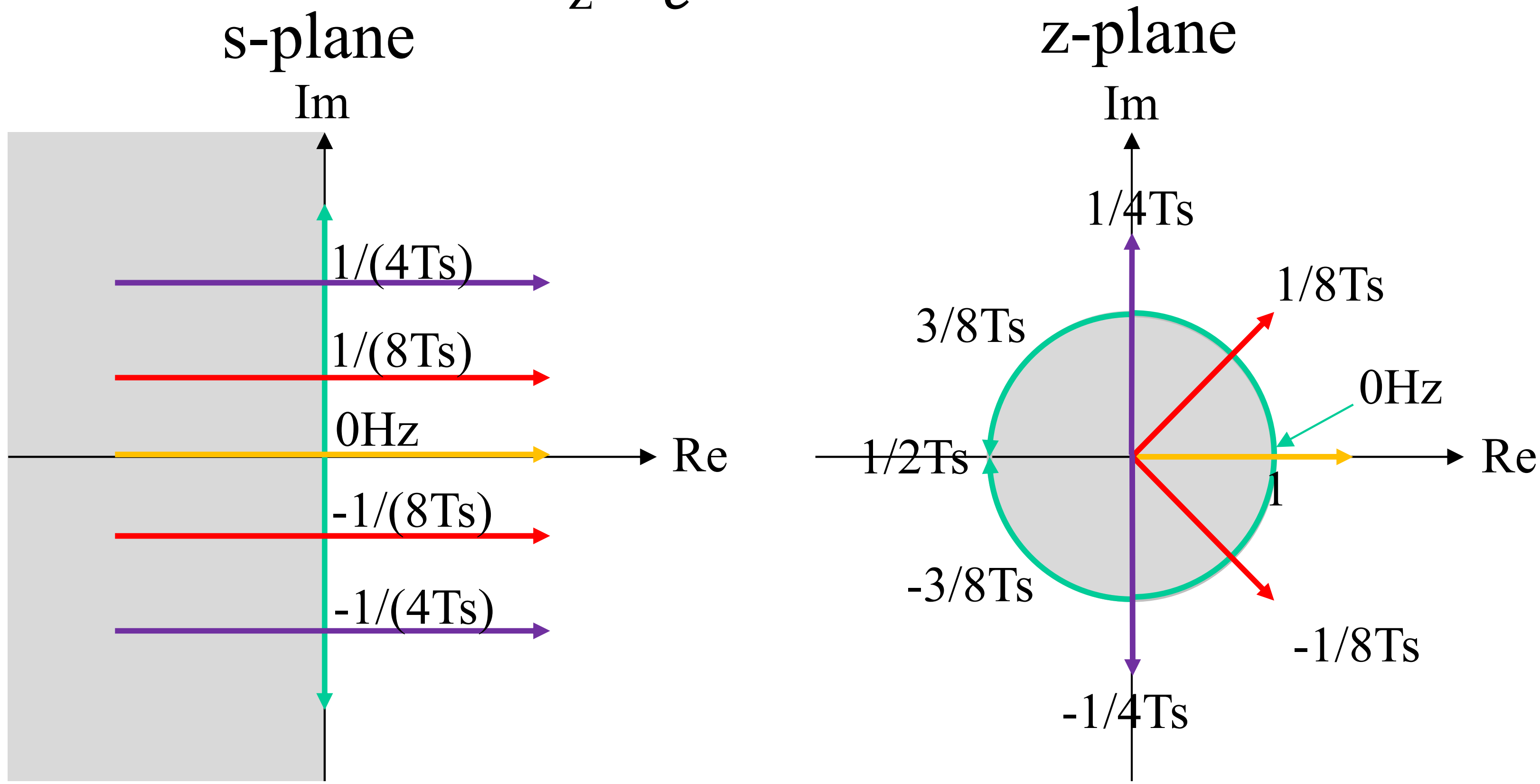
$$z^{-1} = e^{-sT_s} \quad \downarrow$$

$$H(z) = \frac{Output\ signal\ (z)}{Input\ signal\ (z)}$$

The non-linear functions of z-variable are deduced from the rational functions including an integration and a differentiation of s-variable. The complexity of the circuit implementation is avoided by using a translation theorem and some approximations.

Correspondence relation of s-domain and z-domain

$z = e^{(\sigma + j\omega)T_s}$



Time shift in Laplace transform

$k(t) = g(t-T)u(t-T)$, where *u(t)* is a unit step function

$$\Big\downarrow \mathcal{L}$$

$$K(s) = \int_0^\infty g(t-T)u(t-T)e^{-st}dt$$

$$\tau = t - T$$

$$K(s) = \int_{-T}^\infty g(\tau)u(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_0^\infty g(\tau)u(\tau)e^{-s\tau}d\tau$$

$$= e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sT}F(s) \quad \leftarrow \quad f(t) = g(t)u(t)$$

$\Big\downarrow Z$ if $T = nT_s$ (T_s: Sampling period)

$$K(z) = e^{-snT_s}F(z) = z^{-n}F(z)$$

Time shift in Z transform

$$x(nT_s) \xrightarrow{Z} X(z) \qquad \text{, where } t = nT_s \tag{6}$$

(Right shift)
$$x(nT_s - nT_s) \xrightarrow{Z} z^{-n} X(z) \tag{7}$$

(Left shift)
$$x(nT_s + mT_s) \xrightarrow{Z} z^{m} X(z) - z^{m} \sum_{k=0}^{m-1} x(kT_s) z^{-k} \tag{8}$$

$$x_{in}(t) \longrightarrow \boxed{\text{Delay } T_s} \longrightarrow x_{out}(t) \qquad\qquad X_{in}(z) \longrightarrow \boxed{z^{-1}} \longrightarrow X_{out}(z)$$

The equation 7 shows that the multiplication of z^{-n} in z-plane is equivalent to the delay of nT_s in time domain. The z^{-1} operator is called "Delay element".

Approximation of Z transform 1

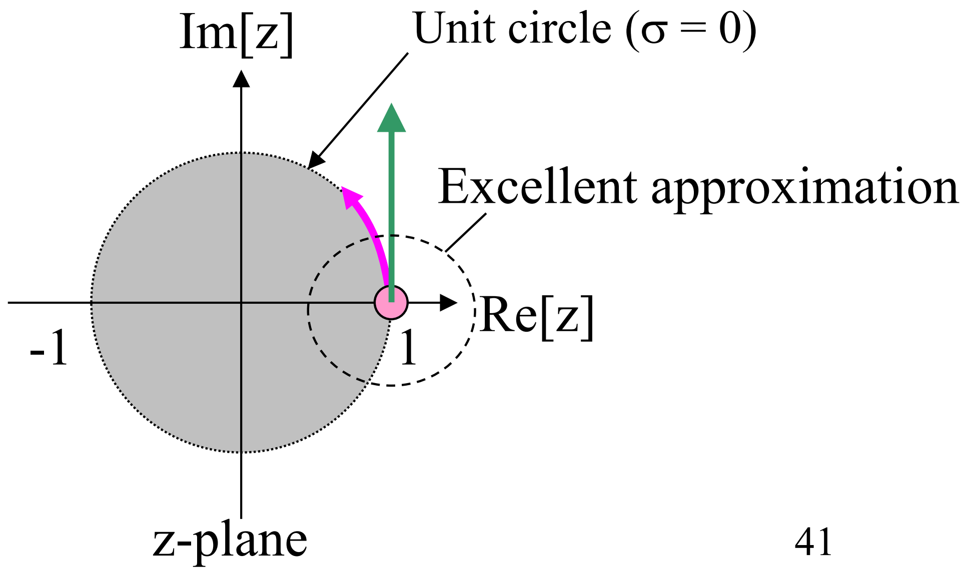
- Forward Euler Transformation (FET)

Power series expansion of z at s = 0

$$z = e^{sT_S} = 1 + \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 + \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 + sT_S$$

$$\therefore s \approx \frac{z-1}{T_S}$$



Approximation of Z transform 2

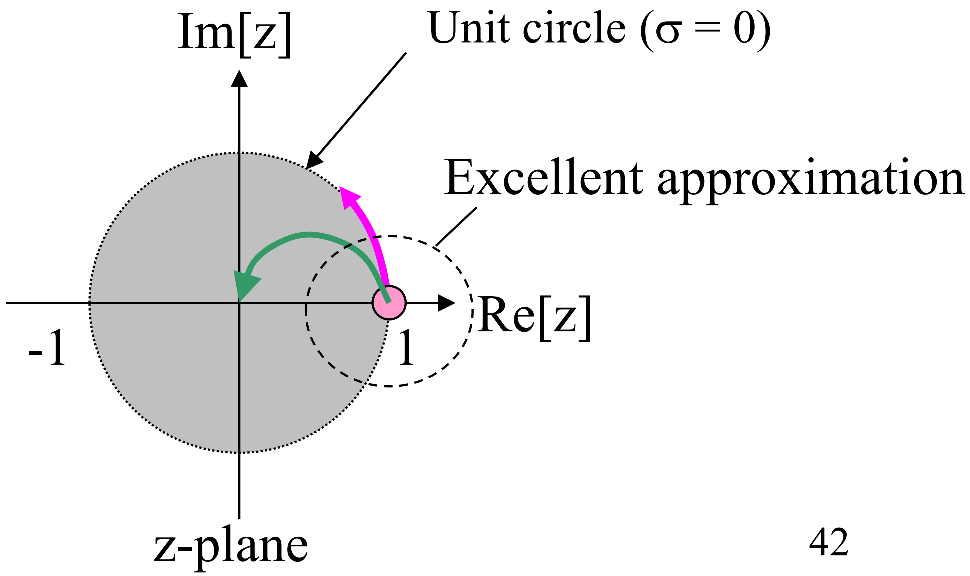
- Backward Euler Transformation (BET)

Power series expansion of z^{-1} at s = 0

$$z^{-1} = e^{-sT_S} = 1 - \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 - \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 - sT_S$$

$$\therefore s \approx \frac{1 - z^{-1}}{T_S}$$



Approximation of Z transform 3

- Bilinear Transformation

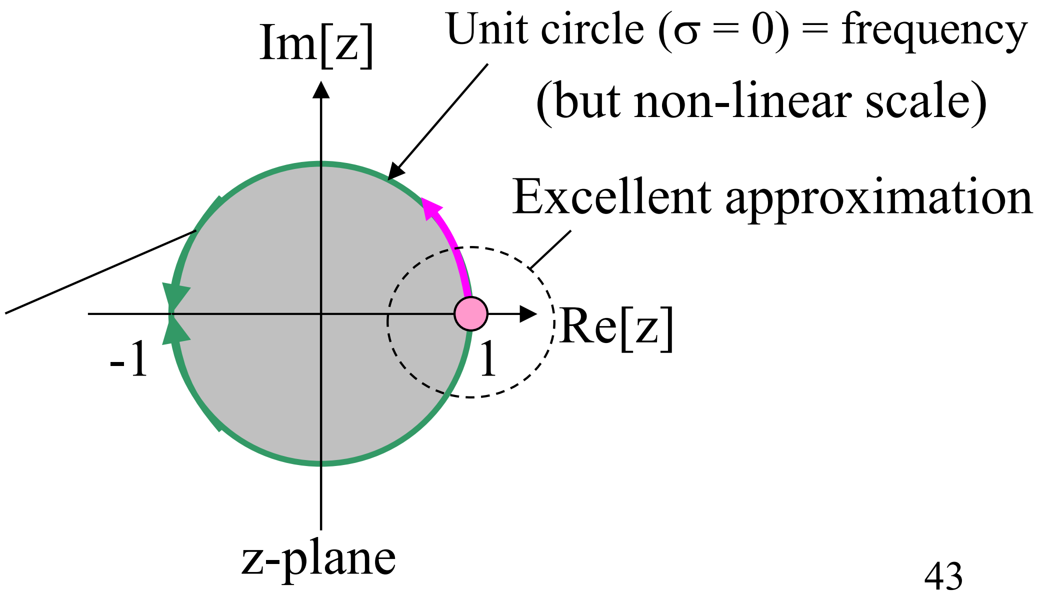
Power series expansion of $\log_e z$ at s = 0

$$sT_S = \ln z = 2\cdot\left[\frac{z-1}{z+1} + \frac{1}{3}\frac{(z-1)^3}{(z+1)^3} + \frac{1}{5}\frac{(z-1)^5}{(z+1)^5} + \Lambda\right]$$

$$\approx 2\frac{z-1}{z+1}$$

$$\therefore s \approx \frac{2}{T_S}\frac{z-1}{z+1} = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\omega_{bilinear} = \frac{2}{T_s} arctan \frac{\omega T_s}{2}$$



Integration of continuous-time signal

$$v_{in}(t) \longrightarrow \boxed{\text{Integrator}} \longrightarrow v_{out}(t)$$

$$v_{out}(t) = \int_0^t v_{in}(\tau)d\tau$$

$\Downarrow$ Laplace transform

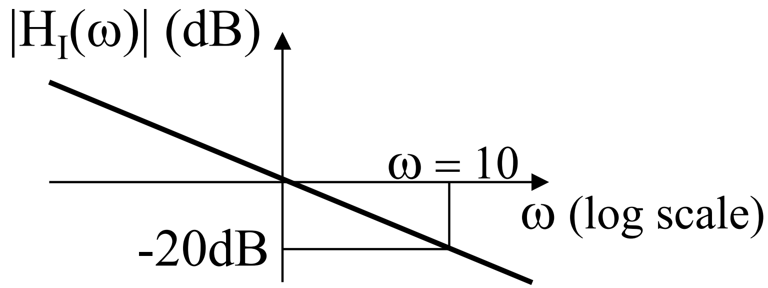
$$V_{out}(s) = \frac{1}{s}V_{in}(s) + \frac{1}{s}\int_{-\infty}^{0} v_{in}(\tau)d\tau$$

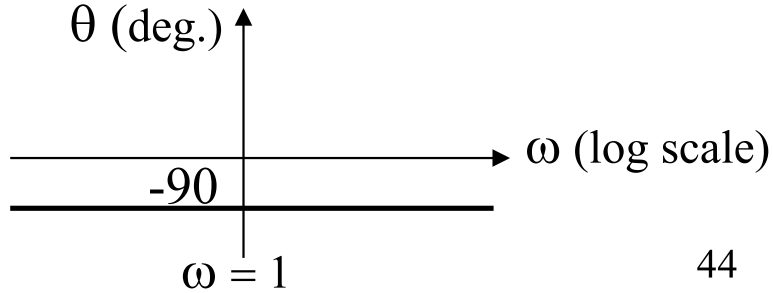
$$= \frac{1}{s}V_{in}(s) \qquad \text{(Periodic function)}$$

Frequency domain transfer function (s = jω)

$$H_I(\omega) = \frac{1}{s} = \frac{1}{j\omega} = -j\frac{1}{\omega}$$

$$|H_I(\omega)|[dB] = 20\log|H_I(\omega)| = 20\log\frac{1}{\omega} = -20\log\omega$$

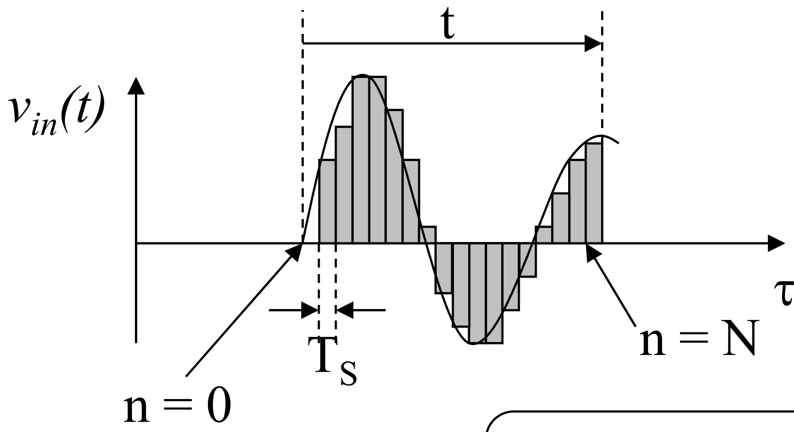




Integration of discrete-time signal

Integration approximated with BET

$$H_I(s) = \frac{1}{s} \cong \frac{T_S}{1 - z^{-1}}$$



$$v_{out}(t) = \int_0^t v_{in}(\tau) d\tau$$

Discretized

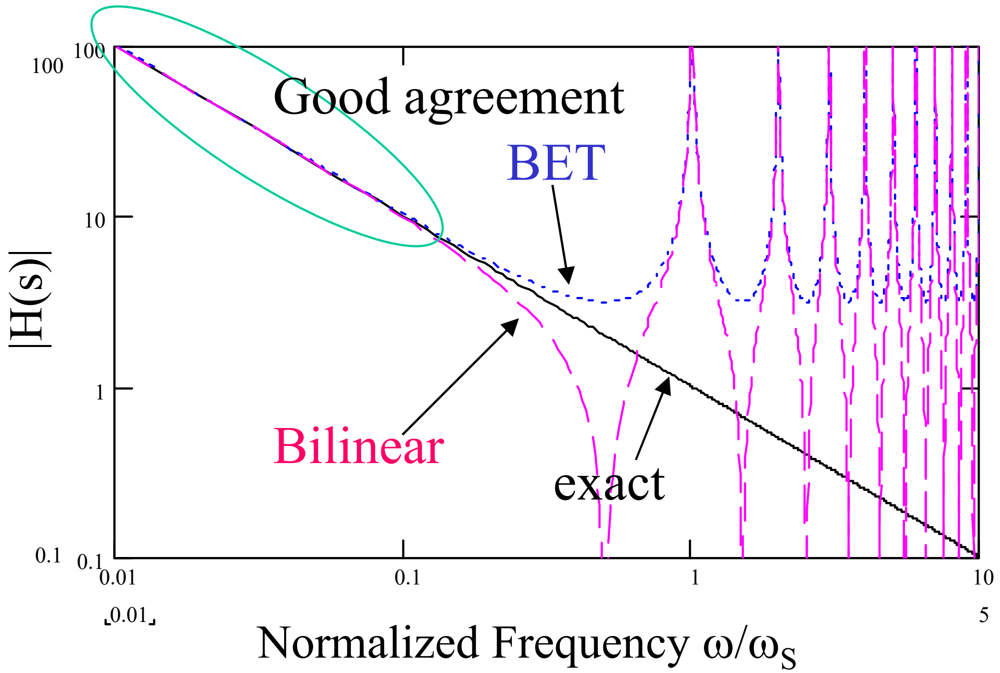
$$v_{out}(t) = \sum_{n=0}^{N} v_{in}(t - nT_S) \cdot T_S \quad \xrightarrow{Z} \quad V_{out}(z) = T_S \sum_{n=0}^{N} z^{-n} V_{in}(z)$$

Geometrical series of z^{-1}

$$\xrightarrow{N=\infty} \underbrace{\frac{T_S}{1 - z^{-1}}}_{H_I(s)} V_{in}(z)$$

Integration error due to approximation

$$\begin{cases} \text{Frequency domain transfer function of integrator} & H(s) = \dfrac{1}{s} = -j\dfrac{1}{\omega} \\ \sigma = 0 \rightarrow \ z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}} \end{cases}$$



BET of Integrator

$$\frac{T_S}{1-z^{-1}} = \frac{T_S}{1-e^{-j\omega T_S}} = \frac{T_S}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

Bilinear transformation of integrator

$$\frac{T_S}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T_S}{2}\frac{1+e^{-j\omega T_S}}{1-e^{-j\omega T_S}} = \frac{T_S}{2}\frac{1+e^{-j2\pi\frac{\omega}{\omega_S}}}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega \ll \omega_s/2$.

Differentiation of continuous-time signal

$v_{in}(t) \longrightarrow$ differentiator $\longrightarrow v_{out}(t)$

$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

⇩ Laplace transform

$$V_{out}(s) = sV_{in}(s) - v_{in}(0)$$

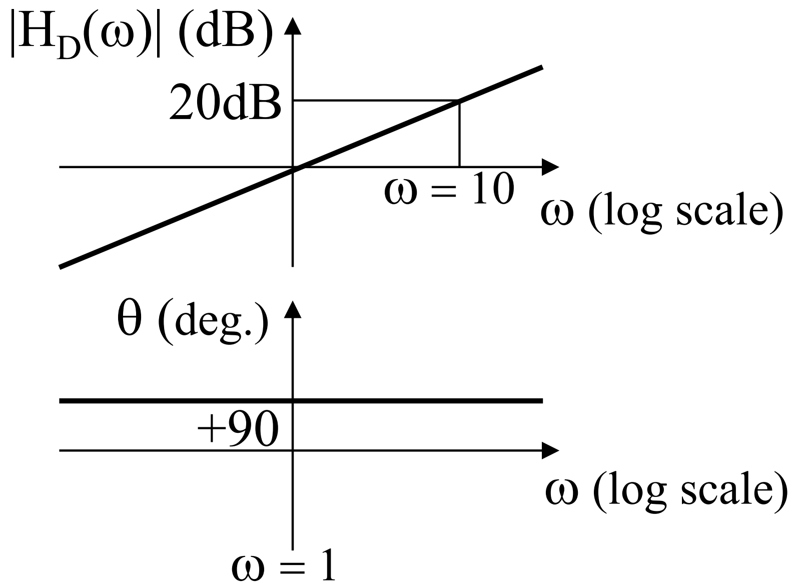
$$V_{out}(s) = sV_{in}(s)$$

（t = 0 で信号がない場合）

Frequency domain transfer function (s = jω)

$$H_D(s) = s = j\omega$$

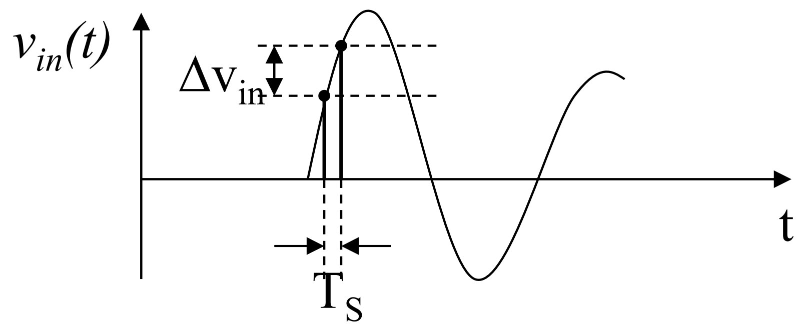
$$|H_D(\omega)|[dB] = 20\log|H_D(\omega)| = 20\log\omega$$



Differentiation of discrete-time signal

Differentiation approximated with BET

$$H_D(s) = s \cong \frac{1 - z^{-1}}{T_S}$$



$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

Discretized

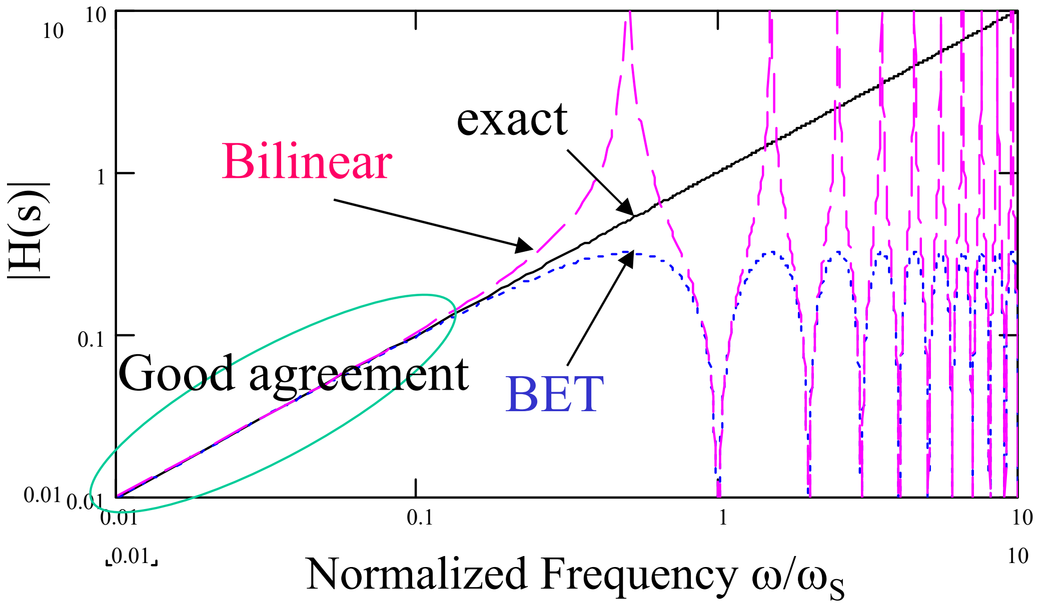
$$v_{out}(t) = \frac{v_{in}(t) - v_{in}(t - T_S)}{T_S} \xrightarrow{Z} V_{out}(z) = \frac{V_{in}(z) - z^{-1} V_{in}(z)}{T_S}$$

Delay element

$$= \underbrace{\frac{1 - z^{-1}}{T_S}}_{H_D(s)} V_{in}(z)$$

Differentiation error due to approximation

$$\begin{cases} \text{Frequency domain transfer function of integrator} \quad H(s) = s = j\omega \\ \sigma = 0 \rightarrow \ z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}} \end{cases}$$



BET of differentiator

$$\frac{1-z^{-1}}{T_S} = \frac{1-e^{-j\omega T_S}}{T_S} = \frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{T_S}$$

Bilinear transformation of differentiator

$$\frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_S}\frac{1-e^{-j\omega T_S}}{1+e^{-j\omega T_S}} = \frac{2}{T_S}\frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{1+e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega << \omega_s/2$.